FRACTIONAL CALCULUS AND SOME PROPERTIES OF SALAGEAN-TYPE K-STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. We define and investigate a new class of Salagean-type k-starlike functions with negative coefficients. We obtain some properties of this subclass. Furthermore, we give Hadamard product of several functions and some distortion theorems for fractional calculus of this generalized class.

Keywords. analytic function, univalent function, k-starlike, Sălăgean operator, Hadamard product.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. For $f(z)$ belongs to $\mathcal{A}$, Salagean [12] has introduced the following operator called the Salagean operator :

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$\vdots$$

$$D^n f(z) = D(D^{n-1} f(z))\ , n \in N_0 = \{0\} \cup \{1, 2, \cdots\}.$$

Let $\mathcal{S}_p$ $(p \geq 1)$ denote the class of functions of the form f$(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ that are holomorphic and p-valent in the unit disk U.

Also, let $\mathcal{A}_p(n)$ denote the subclass of $\mathcal{S}_p$ $(p \geq 1)$ consisting of functions that can be expressed in the form

$$f(z) = z^p - \sum_{m=p+n}^{\infty} a_m z^m \quad ; \quad a_m \geq 0. \tag{1}$$

We can write the following equalities for the functions $f(z)$ belonging to the class $\mathcal{A}_p(n)$:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z) = pz^p - \sum_{m=p+n}^{\infty} m a_m z^m$$

$$\vdots$$

$$D^\lambda f(z) = D(D^{\lambda-1} f(z)) = p^\lambda z^p - \sum_{m=p+n}^{\infty} m^\lambda a_m z^m\ , \lambda \in N_0 = \{0\} \cup N.$$

Denote by $\mathcal{S}$ the class of functions of the form $f(z) = z + a_2 z^2 + ...$, analytic and univalent, by $\mathcal{ST}(\alpha)$ subclass consisting of starlike and univalent of order α and by $k - \mathcal{ST}(0 \leq k < \infty)$ a class of $k-$starlike univalent functions in U, introduced and investigated Lecko and Wisniowska in [1].

It is known that every $f \in k - \mathcal{ST}$ has a continuous extension to $\overline{U}$, $f(U)$ is bounded and $f(\partial U)$ is a rectifiable curve [2].

In the present paper, a subclass $(k, n, p, \alpha, \lambda) - \mathcal{ST}$ of starlike functions in the open unit disk U is introduced. A functions $f(z) \in \mathcal{A}_p(n)$ is said to be in the class $(k, n, p, \alpha, \lambda) - \mathcal{ST}$ if it is satisfies

$$Re\left\{\frac{\zeta}{z} + \frac{(z-\zeta)(D^\lambda f(z))'}{D^\lambda f(z)}\right\} \geq \alpha \tag{2}$$

for some $\alpha\,(0 \leq \alpha < p)$ and $z \in U$.

We note that

i)$(0,1,1,\alpha,0) - \mathcal{ST} \equiv T^{*}(\alpha)$ was introduced by Silverman [3],

ii)$(k,1,1,0,0) - \mathcal{ST} \equiv k - \mathcal{ST}$ was introduced by Lecko and Wisniowska [1],

iii)$(k,1,1,\alpha,0) - \mathcal{ST} \equiv k - \mathcal{ST}(\alpha)$ was studied by Baharati et.al. [5],

iv)$(k,n,1,\alpha,0) - \mathcal{ST} \equiv (k,n,\alpha) - \mathcal{ST}$ was studied by Guney et.al [4],

v)$(k,1,1,\alpha,1) - \mathcal{ST} \equiv (k,1,-1,\alpha) - \mathcal{UCV} \equiv k - \mathcal{UCV}(\alpha)$ was investigated by Baharati et.al [5]. Also, the class $(k,A,B,\alpha) - \mathcal{UCV}$ was introduced by Guney et al.[6].

2.Some results of the class $(k,n,p,\alpha,\lambda) - \mathcal{ST}$

Theorem 2.1 *A function $f \in \mathcal{A}_p(n)$ is in the class $(k,n,p,\alpha,\lambda) - \mathcal{ST}$ iff*

$$\sum_{m=p+n}^{\infty} m^{\lambda}[k(m-1)+m-\alpha]a_m \leq p^{\lambda}(p-\alpha)+p^{\lambda}k(p-1). \tag{3}$$

Proof. Let $(k,n,p,\alpha,\lambda) - \mathcal{ST}$. Then we have from (2)

$$Re\left\{\frac{\zeta}{z}+\frac{(z-\zeta)(D^{\lambda}f(z))'}{D^{\lambda}f(z)}\right\}$$

$$=\mathrm{Re}\left\{\frac{\zeta p^{\lambda}z^{p-1}+p^{\lambda+1}z^{p}-\zeta p^{\lambda+1}z^{p-1}+\zeta\sum\limits_{m=p+n}^{\infty}(m^{\lambda+1}-m^{\lambda})a_m z^{m-1}-\sum\limits_{m=p+n}^{\infty}m^{\lambda+1}a_m z^{m}}{p^{\lambda}z^{p}-\sum\limits_{m=p+n}^{\infty}m^{\lambda}a_m z^{m}}\right\} \geq \alpha.$$

If we choose z and ζ real and $z \to 1^{-}$ and $\zeta \to -k^{+}$, we get,

$$\frac{-kp^{\lambda}+p^{\lambda+1}+kp^{\lambda+1}+k\sum\limits_{m=p+n}^{\infty}(m^{\lambda}-m^{\lambda+1})a_m-\sum\limits_{m=p+n}^{\infty}m^{\lambda+1}a_m}{p^{\lambda}-\sum\limits_{m=p+n}^{\infty}m^{\lambda}a_m} \geq \alpha$$

or

$$\sum_{m=p+n}^{\infty} m^{\lambda}[k(m-1)+m-\alpha]a_m \leq p^{\lambda}(p-\alpha)+p^{\lambda}k(p-1)$$

which is equivalent to (3).

Conversely, assume that (3) is true. Then

$$\left\{\frac{\zeta}{z}+\frac{(z-\zeta)(D^\lambda f(z))'}{D^\lambda f(z)}\right\}=\left\{\frac{\zeta}{z}+\frac{(z-\zeta)\left(p^{\lambda+1}z^{p-1}-\sum\limits_{m=p+n}^{\infty}m^{\lambda+1}a_m z^{m-1}\right)}{p^\lambda z^p-\sum\limits_{m=p+n}^{\infty}m^\lambda a_m z^m}\right\}$$

$$=\left\{\frac{\zeta p^\lambda z^{p-1}+p^{\lambda+1}z^p-\zeta p^{\lambda+1}z^{p-1}+\zeta\sum\limits_{m=p+n}^{\infty}(m^{\lambda+1}-m^\lambda)a_m z^{m-1}-\sum\limits_{m=p+n}^{\infty}m^{\lambda+1}a_m z^m}{p^\lambda z^p-\sum\limits_{m=p+n}^{\infty}m^\lambda a_m z^m}\right\}\geq\alpha.$$

for $|z|<1$. If we choose $z\to 1^-$ and $\zeta\to -k^+$ through real values, we obtain

$$Re\left\{\frac{\zeta}{z}+\frac{(z-\zeta)(D^\lambda f(z))'}{D^\lambda f(z)}\right\}=\frac{p^\lambda k(p-1)+p^{\lambda+1}-\sum\limits_{m=p+n}^{\infty}m^\lambda[k(m-1)+m]a_m}{p^\lambda-\sum\limits_{m=p+n}^{\infty}m^\lambda a_m}. \tag{4}$$

If (3) is rewritten as

$$\sum_{m=p+n}^{\infty}m^\lambda[k(m-1)+m]a_m\leq p^{\lambda+1}-p^\lambda\alpha+p^\lambda k(p-1)+\alpha\sum_{m=p+n}^{\infty}m^\lambda a_m,$$

and (4) is used, then we obtain

$$Re\left\{\frac{\zeta}{z}+\frac{(z-\zeta)(D^\lambda f(z))'}{D^\lambda f(z)}\right\}\geq\frac{\alpha\left(p^\lambda-\sum\limits_{m=p+n}^{\infty}m^\lambda a_m\right)}{p^\lambda-\sum\limits_{m=p+n}^{\infty}m^\lambda a_m}=\alpha.$$

Thus $f\in(k,n,p,\alpha,\lambda)-\mathcal{ST}$.

Theorem 2.2 *If $f\in(k,n,p,\alpha,\lambda)-\mathcal{ST}$, then for $|z|=r<1$ we obtain*

$$r^p-\frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{(p+n)^\lambda[(1+k)(p+n)-k-\alpha]}r^{p+n}\leq|f(z)|\leq$$

$$\leq r^p+\frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{(p+n)^\lambda[(1+k)(p+n)-k-\alpha]}r^{p+n} \tag{5}$$

and

$$pr^{p-1} - \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda-1}[(1+k)(p+n)-k-\alpha]} r^{p+n-1} \leq |f'(z)| \leq$$

$$\leq pr^{p-1} + \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda-1}[(1+k)(p+n)-k-\alpha]} r^{p+n-1}. \tag{6}$$

All inequalities are sharp.

Proof From (3), we have

$$\sum_{m=p+n}^{\infty} a_m \leq \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda}[(1+k)(p+n)-k-\alpha]} \tag{7}$$

and

$$\sum_{m=p+n}^{\infty} m a_m \leq \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda-1}[(1+k)(p+n)-k-\alpha]}. \tag{8}$$

Thus

$$|f(z)| \leq r^p + \sum_{m=p+n}^{\infty} a_m r^m \leq r^p + r^{p+n} \sum_{m=p+n}^{\infty} a_m \leq$$

$$\leq r^p + \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda}[(1+k)(p+n)-k-\alpha]} r^{p+n}$$

and

$$|f(z)| \geq r^p - \sum_{m=p+n}^{\infty} a_m r^m \geq r^p - r^{p+n} \sum_{m=p+n}^{\infty} a_m \geq$$

$$\geq r^p - \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda}[(1+k)(p+n)-k-\alpha]} r^{p+n}.$$

which prove that the assertion (5) of Theorem 2.2.

Furthermore, for $|z| = r < 1$ and (6), we have

$$|f'(z)| \leq pr^{p-1} + \sum_{m=p+n}^{\infty} m a_m r^{m-1} \leq pr^{p-1} + r^{p+n-1} \sum_{m=p+n}^{\infty} m a_m$$

$$\leq pr^{p-1} + \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda-1}[(1+k)(p+n)-k-\alpha]} r^{p+n-1}$$

and

$$|f'(z)| \geq pr^{p-1} - \sum_{m=p+n}^{\infty} m a_m r^{m-1} \geq pr^{p-1} - r^{p+n-1} \sum_{m=p+n}^{\infty} m a_m$$

$$\geq pr^{p-1} - \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda-1}[(1+k)(p+n)-k-\alpha]} r^{p+n-1}$$

which prove that the assertion (6) of Theorem 2.2.

The bounds in (5) and (6) are attained for the function f given by

$$f(z) = z^p - \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda}[(1+k)(p+n)-k-\alpha]} z^{p+n} \tag{9}$$

Theorem 2.3 *Let the functions*

$$f(z) = z^p - \sum_{m=p+n}^{\infty} a_m z^m; a_m \geq 0.$$

and

$$g(z) = z^p - \sum_{m=p+n}^{\infty} b_m z^m; b_m \geq 0.$$

be in the class $(k, n, p, \alpha, \lambda) - \mathcal{ST}$. *Then for* $0 \leq \rho \leq 1$,

$$h(z) = (1-\rho)f(z) + \rho\, g(z) = z^p - \sum_{m=p+n}^{\infty} c_m z^m; c_m \geq 0.$$

is in the class $(k, n, p, \alpha, \lambda) - \mathcal{ST}$.

Proof. Assume that $f, g \in (k, n, p, \alpha, \lambda) - \mathcal{ST}$.

Then we have from Theorem 2.1

$$\sum_{m=p+n}^{\infty} m^{\lambda}[k(m-1)+m-\alpha] a_m \leq p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)$$

and

$$\sum_{m=p+n}^{\infty} m^{\lambda}[k(m-1)+m-\alpha]b_m \leq p^{\lambda}(p-\alpha)+p^{\lambda}k(p-1).$$

Therefore we can see that

$$\sum_{m=p+n}^{\infty} m^{\lambda}[k(m-1)+m-\alpha]c_m =$$

$$= \sum_{m=p+n}^{\infty} m^{\lambda}[k(m-1)+m-\alpha][(1-\rho)a_m+\rho b_m] \leq p^{\lambda}(p-\alpha)+p^{\lambda}k(p-1)$$

which completes the proof of Theorem 2.3.

Definition 2.1 *The Modified Hadamard Product $f * g$ of two functions*

$$f(z) = z^p - \sum_{m=p+n}^{\infty} a_m z^m; a_m \geq 0$$

and

$$g(z) = z^p - \sum_{m=p+n}^{\infty} b_m z^m; b_m \geq 0$$

are denoted by

$$(f * g)(z) = z^p - \sum_{m=p+n}^{\infty} a_m b_m z^m.$$

Theorem 2.4 *If $f, g \in (k, n, p, \alpha, \lambda)-\mathcal{ST}$ then $f*g \in (k, n, p, \beta, \lambda)-\mathcal{ST}$, where*

$$\beta = \beta(k, n, p, \alpha, \lambda) =$$

$$= \frac{(p^{\lambda+1}+p^{\lambda}k(p-1))(p+n)^{\lambda}[(1+k)(p+n)-k-\alpha]^2}{p^{\lambda}(p+n)^{\lambda}[(1+k)(p+n)-k-\alpha]^2-(p^{\lambda}(p-\alpha)+p^{\lambda}k(p-1))^2}$$

$$-\frac{(p^{\lambda}(p-\alpha)+p^{\lambda}k(p-1))^2[(1+k)(p+n)-k]}{p^{\lambda}(p+n)^{\lambda}[(1+k)(p+n)-k-\alpha]^2-(p^{\lambda}(p-\alpha)+p^{\lambda}k(p-1))^2}.$$

The result is sharp for $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = z^p - \frac{p^\lambda(p-\alpha) + p^\lambda k(p-1)}{(p+n)^\lambda[(1+k)(p+n) - k - \alpha]} z^{p+n}$$

where $0 \leq \alpha < p$ *and* $0 \leq k < \infty$.

Proof. From Theorem 2.1, we have

$$\sum_{m=p+n}^{\infty} \frac{m^\lambda[k(m-1) + m - \alpha]}{p^\lambda(p-\alpha) + p^\lambda k(p-1)} a_m \leq 1 \tag{10}$$

and

$$\sum_{m=p+n}^{\infty} \frac{m^\lambda[k(m-1) + m - \alpha]}{p^\lambda(p-\alpha) + p^\lambda k(p-1)} b_m \leq 1. \tag{11}$$

We have to find the largest β such that

$$\sum_{m=p+n}^{\infty} \frac{m^\lambda[k(m-1) + m - \beta]}{p^\lambda(p-\beta) + p^\lambda k(p-1)} a_m b_m \leq 1. \tag{12}$$

From (10) and (11) we obtain, by means of Cauchy-Schwarz inequality,that

$$\sum_{m=p+n}^{\infty} \frac{m^\lambda[k(m-1) + m - \alpha]}{p^\lambda(p-\alpha) + p^\lambda k(p-1)} \sqrt{a_m b_m} \leq 1. \tag{13}$$

Therefore, (12) is true if

$$\frac{m^\lambda[k(m-1) + m - \beta]}{p^\lambda(p-\beta) + p^\lambda k(p-1)} a_m b_m \leq \frac{m^\lambda[k(m-1) + m - \alpha]}{p^\lambda(p-\alpha) + p^\lambda k(p-1)} \sqrt{a_m b_m}$$

or

$$\sqrt{a_m b_m} \leq \frac{[p^\lambda(p-\beta) + p^\lambda k(p-1)][k(m-1) + m - \alpha]}{[p^\lambda(p-\alpha) + p^\lambda k(p-1)][k(m-1) + m - \beta]}.$$

Note that from (13)

$$\sqrt{a_m b_m} \leq \frac{[p^\lambda(p-\alpha) + p^\lambda k(p-1)]}{m^\lambda[k(m-1) + m - \alpha]}.$$

Thus if

$$\frac{[p^\lambda(p-\alpha)+p^\lambda k(p-1)]}{m^\lambda[k(m-1)+m-\alpha]} \leq \frac{[p^\lambda(p-\beta)+p^\lambda k(p-1)][k(m-1)+m-\alpha]}{[p^\lambda(p-\alpha)+p^\lambda k(p-1)][k(m-1)+m-\beta]}$$

or, equivalently, if

$$\beta \leq \frac{(p^{\lambda+1}+p^\lambda k(p-1))m^\lambda[(1+k)m-k-\alpha]^2}{p^\lambda m^\lambda[(1+k)m-k-\alpha]^2-(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2}$$

$$-\frac{(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2[(1+k)m-k]}{p^\lambda m^\lambda[(1+k)m-k-\alpha]^2-(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2}$$

then (12) satisfied. Defining the function $\Phi(m)$ by

$$\Phi(m) = \frac{(p^{\lambda+1}+p^\lambda k(p-1))m^\lambda[(1+k)m-k-\alpha]^2}{p^\lambda m^\lambda[(1+k)m-k-\alpha]^2-(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2}$$

$$-\frac{(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2[(1+k)m-k]}{p^\lambda m^\lambda[(1+k)m-k-\alpha]^2-(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2}.$$

We can see that $\Phi(m)$ is increasing function of m. Therefore,

$$\beta \leq \Phi(p+n) = \frac{(p^{\lambda+1}+p^\lambda k(p-1))(p+n)^\lambda[(1+k)(p+n)-k-\alpha]^2}{p^\lambda(p+n)^\lambda[(1+k)(p+n)-k-\alpha]^2-(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2}$$

$$-\frac{(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2[(1+k)(p+n)-k]}{p^\lambda(p+n)^\lambda[(1+k)(p+n)-k-\alpha]^2-(p^\lambda(p-\alpha)+p^\lambda k(p-1))^2}.$$

which completes the assertion of theorem.

3. Extreme Points for $(k, n, p, \alpha, \lambda) - \mathcal{ST}$

Theorem 3.1 *Let $f_{p+n-1}(z) = z^p$ and $f_m(z) = z^p - \frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{m^\lambda[k(m-1)+m-\alpha]} z^m$, $m = p+n, p+n+1, \ldots$. Then If $f \in (k, n, p, \alpha, \lambda) - \mathcal{ST}$ iff it can be expressed in the form*

$$f(z) = \sum_{m=p+n+1}^{\infty} \xi_m f_m(z),$$

where $\xi_m \geq 0$ *and* $\sum\limits_{m=p+n-1}^{\infty} \xi_m = 1.$

Proof. Assume that

$$f(z) = \sum_{m=p+n-1}^{\infty} \xi_m f_m(z).$$

Then

$$f(z) = \xi_{p+n-1} f_{p+n-1}(z) + \sum_{m=p+n}^{\infty} \xi_m f_m(z) =$$

$$= \xi_{p+n-1} z^p + \sum_{m=p+n}^{\infty} \xi_m \left[z^p - \frac{p^\lambda (p-\alpha) + p^\lambda k(p-1)}{m^\lambda [k(m-1) + m - \alpha]} z^m \right]$$

$$= \left(\sum_{m=p+n-1}^{\infty} \xi_m \right) z^p - \sum_{m=p+n}^{\infty} \xi_m \frac{p^\lambda (p-\alpha) + p^\lambda k(p-1)}{m^\lambda [k(m-1) + m - \alpha]} z^m$$

$$= z^p - \sum_{m=p+n}^{\infty} \xi_m \frac{p^\lambda (p-\alpha) + p^\lambda k(p-1)}{m^\lambda [k(m-1) + m - \alpha]} z^m.$$

Thus

$$\sum_{m=p+n}^{\infty} \xi_m \left(\frac{p^\lambda (p-\alpha) + p^\lambda k(p-1)}{m^\lambda [k(m-1) + m - \alpha]} \right) \left(\frac{m^\lambda [k(m-1) + m - \alpha]}{p^\lambda (p-\alpha) + p^\lambda k(p-1)} \right) = \sum_{m=p+n}^{\infty} \xi_m$$

$$= \sum_{m=p+n-1}^{\infty} \xi_m - \xi_{p+n-1} = 1 - \xi_m \leq 1.$$

Hence $f \in (k, n, p, \alpha, \lambda) - \mathcal{ST}$.

Conversely, suppose that $f \in (k, n, p, \alpha, \lambda) - \mathcal{ST}$. Since

$$|a_m| \leq \frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{m^\lambda[k(m-1)+m-\alpha]} \quad m = p+n, p+n+1, ...,$$

we can set

$$\xi_m = \frac{m^\lambda[k(m-1)+m-\alpha]}{p^\lambda(p-\alpha)+p^\lambda k(p-1)} a_m, \quad m = p+n, p+n+1, ...$$

and

$$\xi_{p+n-1} = 1 - \sum_{m=p+n}^{\infty} \xi_m.$$

Then

$$f(z) = z^p - \sum_{m=p+n}^{\infty} a_m z^m = z^p - \sum_{m=p+n}^{\infty} \frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{m^\lambda[k(m-1)+m-\alpha]} \xi_m z^m$$

$$= z^p - \sum_{m=p+n}^{\infty} \xi_m(z^p - f_m(z)) = \left(1 - \sum_{m=p+n}^{\infty} \xi_m\right) z^p + \sum_{m=p+n}^{\infty} \xi_m f_m(z)$$

$$= \xi_{n+p-1} z^p + \sum_{m=p+n}^{\infty} \xi_m f_m(z) = \xi_{n+p-1} f_{n+p-1}(z) + \sum_{m=p+n}^{\infty} \xi_m f_m(z)$$

$$= \sum_{m=p+n-1}^{\infty} \xi_m f_m(z).$$

This completes the assertion of theorem.

Corollary 3.1 *The extreme points of $(k, n, p, \alpha, \lambda) - \mathcal{ST}$ are given by*

$$f_{p+n-1}(z) = z^p$$

and

$$f_m(z) = z^p - \frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{m^\lambda[k(m-1)+m-\alpha]} z^m, \quad m = p+n, p+n+1, ...$$

4. Definitions and Applications of the Fractional Calculus

In this section, we shall prove several distortion theorems for functions to general class $(k, n, p, \alpha, \lambda)-\mathcal{ST}$. Each of these theorems would involve certain operators of fractional calculus we find it to be convenient to recall here the following definition which were used recently by Owa [7] (and more recently, by Owa and Srivastava [3], and Srivastava and Owa [8] and Srivastava and Owa [9] ; see also Srivastava et all. [10])

Definition 4.1. *The fractional integral of order λ is defined, for a function f , by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta \qquad (\mu > 0) \tag{14}$$

where f is an analytic function in a simply connected region of the z -plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 4.2. *The fractional derivative of order μ is defined, for a function f, by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta (0 \le \mu < 1) \tag{15}$$

where f is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 4.3. *Under the hypotheses of Definition 2, the fractional derivative of order $(n+\mu)$ is defined by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \qquad (0 \le \mu < 1) \tag{16}$$

where $0 \le \mu < 1$ and $n \in IN_0 = IN \bigcup \{0\}$.

From Definition 4.2, we have

$$D_z^0 f(z) = f(z) \tag{17}$$

which, in view of Definition 4.3 yields,

$$D_z^{n+0} f(z) = \frac{d^n}{d\, z^n} D_z^0 f(z) = f^n(z). \tag{18}$$

Thus, it follows from (17) and (18) that

$$\lim_{\mu \to 0} D_z^{-\mu} f(z) = f(z)$$

and

$$\lim_{\mu \to 0} D_z^{1-\mu} f(z) = f'(z).$$

Theorem 4.1. *Let the function f defined by*

$$f(z) = z^p - \sum_{m=p+n}^{\infty} a_m z^m \quad ; \quad a_m \geq 0$$

be in the class $(k, n, p, \alpha, \lambda) - \mathcal{ST}$. Then

$$\left| D_z^{-\mu}(D^i f(z)) \right| \geq |z|^{p+\mu} \cdot$$

$$\cdot \left\{ \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \frac{\Gamma(p+n+1)}{\Gamma(p+n+\mu+1)} \frac{p^\lambda(p-\alpha) + p^\lambda k(p-1)}{(p+n)^{\lambda-i}[(1+k)(p+n)-k-\alpha]} |z|^n \right\} \tag{19}$$

and

$$\left| D_z^{-\mu}(D^i f(z)) \right| \leq |z|^{p+\mu} \cdot$$

$$\cdot \left\{ \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + \frac{\Gamma(p+n+1)}{\Gamma(p+n+\mu+1)} \frac{p^\lambda(p-\alpha) + p^\lambda k(p-1)}{(p+n)^{\lambda-i}[(1+k)(p+n)-k-\alpha]} |z|^n \right\} \tag{20}$$

for $\mu > 0$, $0 \leq i \leq \lambda$ and $z \in U$. The equalities in (19) and (20) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{p^\lambda(p-\alpha) + p^\lambda k(p-1)}{[(1+k)(p+n)-k-\alpha](p+n)^\lambda} z^{p+n}. \tag{21}$$

Proof. We note that

$$\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu}(D^i f(z)) = z^p - \sum_{m=p+n}^{\infty} \frac{\Gamma(m+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(m+\mu+1)} m^i a_m z^m.$$

Defining the function $\varphi(m)$

$$\varphi(m) = \frac{\Gamma(m+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(m+\mu+1)}; \mu > 0.$$

We can see that $\varphi(m)$ is decreasing in m, that is

$$0 < \varphi(m) \leq \varphi(p+n) = \frac{\Gamma(p+n+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+n+\mu+1)}.$$

On other hand, from [11]

$$\sum_{m=p+n}^{\infty} m^i a_m \leq \frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{[(1+k)(p+n)-k-\alpha]}(p+n)^{-(\lambda-i)}; \qquad 0 \leq i \leq \lambda.$$

Therefore,

$$\left|\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu}(D^i f(z))\right| \geq |z|^p - \varphi(p+n)|z|^{p+n} \sum_{m=p+n}^{\infty} m^i a_m$$

$$\geq |z|^p - \frac{\Gamma(p+n+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+n+\mu+1)} \frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{m^\lambda[k(m-1)+m-\alpha]}(p+n)^{-(\lambda-i)}|z|^{p+n}$$

and

$$\left|\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu}(D^i f(z))\right| \leq |z|^p + \varphi(p+n)|z|^{p+n} \sum_{m=p+n}^{\infty} m^i a_m$$

$$\leq |z|^p + \frac{\Gamma(p+n+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+n+\mu+1)} \frac{p^\lambda(p-\alpha)+p^\lambda k(p-1)}{m^\lambda[k(m-1)+m-\alpha]}(p+n)^{-(\lambda-i)}|z|^{p+n}.$$

which completes the proof of theorem.

Next, we prove

Theorem 4.2. *Let the function f defined by*

$$f(z) = z^p - \sum_{m=p+n}^{\infty} a_m z^m; \quad a_m \geq 0$$

be in the class $(k, n, p, \alpha, \lambda) - \mathcal{ST}$. *Then*

$$\left| D_z^{\mu}(D^i f(z)) \right| \geq |z|^{p-\mu} \cdot$$

$$\cdot \left\{ \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \frac{\Gamma(p+n)}{\Gamma(p+n-\mu+1)} \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda-i-1}[(1+k)(p+n)-k-\alpha]} |z|^n \right\} \tag{22}$$

and

$$\left| D_z^{\mu}(D^i f(z)) \right| \leq |z|^{p-\mu} \cdot$$

$$\cdot \left\{ \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \frac{\Gamma(p+n)}{\Gamma(p+n-\mu+1)} \frac{p^{\lambda}(p-\alpha) + p^{\lambda}k(p-1)}{(p+n)^{\lambda-i-1}[(1+k)(p+n)-k-\alpha]} |z|^n \right\} \tag{23}$$

for $0 \leq \mu < 1$, $0 \leq i \leq \lambda - 1$ *and* $z \in U$. *The equalities (22) and (23) are attained for the function* $f(z)$ *given by (21).*

Proof. Using similar argument as given by Theorem 4.1, we can get result.

Remark. *Letting* $\mu \to 0$ *and* $i = 0$ *in Theorem 4.1 and taking* $\mu \to 1$ *and* $i = 0$ *in Theorem 4.2, we obtain the inequalities (5) and (6) in Theorem 2.2, respectively.*

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