UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

Virgil Pescar

Abstract. In this work some integral operators are studied and the author determines conditions for the univalence of these integral operators.

Key words : integral operator, univalence.

2000 *Mathematics Subject Classification.* Primary 30C45.

1. Introduction

Let $U = \{z \in C : |z| < 1\}$ be the unit disc in the complex plane and let A be the class of functions which are analytic in the unit disk normalized with $f(0) = f'(0) - 1 = 0$.

Let S the class of the functions $f \in A$ which are univalent in U.

2. Preliminary results

In order to prove our main results we will use the theorems presented in this section.

Theorem 2.1.[3]. *Assume that $f \in A$ satisfies condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \; z \in U, \tag{1}$$

then f is univalent in U.

THEOREM 2.2.[4]. *Let α be a complex number, $Re\alpha > 0$ and $f(z) = z + a_2z^2 + \dots$ is a regular function in U. If*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2}$$

for all $z \in U$, then for any complex number β, $Re\beta \geq Re\alpha$ the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} = z + \dots \tag{3}$$

is regular and univalent in U.

SCHWARZ LEMMA [1]. *Let $f(z)$ the function regular in the disk $U_R = \{z \in C; |z| < R\}$, with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| < \frac{M}{R^m}|z|^m, \ \ z \in U_R \tag{4}$$

the equality (in the inequality (4) for $z \neq 0$) can hold only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

3.Main results

THEOREM 3.1. *Let $g \in A, \gamma$ be a complex number such that $Re\,\gamma \geq 1$, M be a real number and $M > 1$.*

If

$$|zg'(z)| < M, \ z \in U \tag{5}$$

and

$$|\gamma| \leq \frac{3\sqrt{3}}{2M} \tag{6}$$

then the function

$$T_\gamma(z) = \left[\gamma\int_0^z u^{\gamma-1}\left(e^{g(u)}\right)^\gamma du\right]^{\frac{1}{\gamma}} \tag{7}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left(e^{g(u)}\right)^\gamma du \tag{8}$$

which is regular in U.

The function

$$h(z) = \frac{1}{|\gamma|} \frac{zf''(z)}{f'(z)} \tag{9}$$

where the constant $|\gamma|$ satisfies the inequality (6), is regular in U.

From (9) and (8) it follows that

$$h(z) = \frac{\gamma}{|\gamma|} zg'(z). \tag{10}$$

Using (10) and (5) we have

$$|h(z)| < M \tag{11}$$

for all $z \in U$. From (10) we obtain $h(0) = 0$ and applying Schwarz-Lemma we obtain

$$\frac{1}{|\gamma|} \left| \frac{zf''(z)}{f'(z)} \right| \leq M|z| \tag{12}$$

for all $z \in U$, and hence, we obtain

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| M |z| \left(1 - |z|^2\right). \tag{13}$$

Let us consider the function $Q : [0,1] \to \Re$, $Q(x) = x\left(1 - x^2\right)$, $x = |z|$. We have

$$Q(x) \leq \frac{2}{3\sqrt{3}} \tag{14}$$

for all $x \in [0,1]$. From (14), (13) and (6) we obtain

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1. \tag{15}$$

for all $z \in U$. From (8) we obtain $f'(z) = \left(e^{g(z)}\right)^\gamma$. Then, from (15) and Theorem 2.2 for $Re\alpha = 1$ it follows that the function T_γ is in the class S.

Theorem 3.2. *Let $g \in A$, satisfy (1), γ be a complex number with $Re\,\gamma \geq 1$, M be a real number, $M > 1$ and $|\gamma - 1| \leq \frac{54\,M^4}{(12M^4+1)\sqrt{12\,M^4+1}+36\,M^4-1}$. If*

$$|g(z)| < M, \;\; z \in U, \tag{16}$$

then the function

$$H_\gamma(z) = \left[\gamma \int_0^z u^{2\gamma-2} \left[e^{g(u)}\right)^{\gamma-1} du\right]^{\frac{1}{\gamma}} \tag{17}$$

is in the class S.

Proof. We observe that

$$H_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1} \left(ue^{g(u)}\right)^{\gamma-1} du\right]^{\frac{1}{\gamma}}. \tag{18}$$

Let us consider the function

$$h(z) = \int_0^z \left(ue^{g(u)}\right)^{\gamma-1} du. \tag{19}$$

The function h is regular in U.
From (19) we obtain

$$\frac{h''(z)}{h'(z)} = (\gamma - 1)\,\frac{zg'(z)+1}{z} \tag{20}$$

and hence, we have

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right| = |\gamma-1|\left(1-|z|^2\right)|zg'(z)+1| \tag{21}$$

for all $z \in U$. From (21) we get

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right| \leq |\gamma-1|\left(1-|z|^2\right)\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\frac{|g^2(z)|}{|z|}+1\right) \tag{22}$$

for all $z \in U$.

By the Schwarz Lemma also $|g(z)| \leq M|z|$, $z \in U$ and using (22) we obtain

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right| \leq |\gamma-1|\left(1-|z|^2\right)\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|M^2|z|+M^2|z|+1\right) \tag{23}$$

for all $z \in U$.

Since g satisfies the condition (1) then from (23) we have

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right| \leq |\gamma-1|\left(1-|z|^2\right)\left(2M^2|z|+1\right) \tag{24}$$

for all $z \in U$.

Let us consider the function $G : [0,1] \to \Re$, $G(x) = (1-x^2)\,(2M^2x + 1)$, $x = |z|$.

We have

$$G(x) \leq \frac{(12\,M^4+1)\sqrt{12\,M^4+1}+36\,M^4-1}{54\,M^4} \tag{25}$$

for all $x \in [0,1]$.

Since $|\gamma-1| \leq \frac{54\,M^4}{(12\,M^4+1)\sqrt{12\,M^4+1}+36\,M^4-1}$, from (25) and (24) we conclude that

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right| \leq 1. \tag{26}$$

for all $z \in U$.

Now (26) and Theorem 2.2 for $Re\,\alpha = 1$ imply that the function H_γ is in the class S.

Remark. *For* $0 < M \leq 1$, *Theorem 3.1 and Theorem 3.2 hold only in the case* $g(z) = Kz$, *where* $|K| = 1$.

References

[1] O. Mayer, *The functions theory of one variable complex*, Bucureşti, 1981.

[2] Z. Nehari, *Conformal mapping*, Mc Graw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975)

[3] S. Ozaki, M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. 33(2), 1972, 392-394.

[4] N. N. Pascu, *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilov, Braşov, (1987), 43-48.

[5] V. Pescar, *New univalence criteria*,"Transilvania" University of Braşov, Braşov, 2002.

[6] C. Pommerenke, *Univalent functions*, Gottingen, 1975.

Author:
Virgil Pescar
"Transilvania" University of Braşov
Faculty of Mathematics and Computer Science
Department of Mathematics
2200 Braşov Romania
e-mali: virgilpescar@unitbv.ro