NEW UNIVALENCE CONDITIONS FOR AN INTEGRAL OPERATOR OF THE CLASS $S(P)$ AND T_2

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Abstract.In this paper we present a few conditions of univalence for the operator $F_{\alpha,\beta}$ on the classes of univalent functions $S(p)$ and T_2. These are actually generalizations(extensions) of certain results published in the papers [1] and [2].

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1.Introduction

We present a few aspects related to the classes of functions, $S(p)$ and T_2.

Let be the class of analytical functions, $A = \{f : f = z + a_2z^2 + ...\}, z \in U$, where U is the unit disk , $U = \{z : |z| < 1\}$. We denote by S, the class of univalent functions on the unit disk.

Let p be a real number with the property $0 < p \leq 2$. We define the class $S(p)$ as the class of functions $f \in A$, which satisfy the conditions $f(z) \neq 0$ and $\left|(z/f(z))''\right| \leq p, z \in U$. Also if $f \in S(p)$ then the following property is true

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq p\,|z|^2, z \in U,$$

relation proved in [5].

We denote by T_2 the class of the univalent functions that satisfy the condition

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, z \in U,$$

and also have the property $f''(0) = 0$.

These functions have the form $f = z + a_3 z^3 + a_4 z^4 + ...$ For $0 < \mu < 1$ we have a subclass of functions denoted by $T_{\mu,2}$, containing the functions $f \in T_2$ that satisfy the property

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < \mu < 1, z \in U.$$

Next we present some well known results related to these classes, results on which we shall rely in this paper.

THE SCHWARTZ LEMMA. *Let the analytic function g be a regular function on the unit disk U and $g(0) = 0$. If $|g(z)| \le 1, \forall z \in U$, then*

$$|g(z)| \le |z|, \forall z \in U \tag{1}$$

and equality holds if and only if $g(z) = \varepsilon z$, where $|\varepsilon| = 1$.

THEOREM 1.[3]. *Let $\alpha \in \mathbf{C}, \mathrm{Re}\alpha > 0$ and $f \in A$. If*

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \le 1, \forall z \in U \tag{2}$$

then $\forall \beta \in \mathbf{C}, \mathrm{Re}\beta \ge \mathrm{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)\, dt \right]^{1/\beta} \tag{3}$$

is univalent.

In his paper [4] Pescar proved the following result:

THEOREM 2.*Assume that $g \in A$ satisfies the condition $\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| < 1, z \in U$, and α is a complex number with*

$$|\alpha - 1| \le \frac{\mathrm{Re}\,\alpha}{3}. \tag{4}$$

If

$$|g(z)| \le 1, \forall z \in U \tag{5}$$

then the function

$$G_{\alpha}(z)=\left(\alpha\int_{0}^{z}g^{(\alpha-1)}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \tag{6}$$

is univalent.

2.MAIN RESULTS

THEOREM 3. *Let be* $g_i \in T_2, g_i(z)=z+a_3^i z^3+a_4^i z^4+..., \forall i=\overline{1,n}, n\in N^*$, *which satisfy the properties*

$$\left|\frac{z^2 g_i^{'}(z)}{g_i^2(z)}-1\right|<1, \forall z\in U, \forall i=\overline{1,n}. \tag{7}$$

If $|g_i(z)|\leq 1,\ \forall z\in U,\ \forall i=\overline{1,n}$, *then for any complex number* α, *satisfying the properties*

$$\mathrm{Re}\alpha>0,\ \mathrm{Re}\left(n(\alpha-1)+1\right)\geq Re\alpha,\ and\ |\alpha-1|\leq\frac{\mathrm{Re}\,\alpha}{3n}. \tag{8}$$

the function

$$F_{\alpha,n}(z)=\left((n(\alpha-1)+1)\int_{0}^{z}g_1^{\alpha-1}(t)\ldots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}} \tag{9}$$

is univalent.

Proof. From (9), $F_{\alpha,n}$ can be written as

$$F_{\alpha,n}(z)=\left((n(\alpha-1)+1)\int_{0}^{z}t^{n(\alpha-1)}\left(\frac{g_1(t)}{t}\right)^{\alpha-1}\ldots\left(\frac{g_n(t)}{t}\right)^{\alpha-1}\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}}. \tag{10}$$

Let us consider the function

$$f(z)=\int_0^z\left(\frac{g_1(t)}{t}\right)^{\alpha-1}\cdots\left(\frac{g_n(t)}{t}\right)^{\alpha-1}\mathrm{d}t. \tag{11}$$

The function f is regular in U, and from (11) we obtain

$$f'(z)=\left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1} \tag{12}$$

and

$$f''(z)=E_1f'(z)\frac{z}{g_1(z)}+\cdots+E_nf'(z)\frac{z}{g_1(z)} \tag{13}$$

where, $E_k=(\alpha-1)\frac{zg_k'(z)-g_k(z)}{z^2}, \forall k=\overline{1,n}.$

Next we calculate the expression $\frac{zf''}{f'}$.

$$\frac{zf''(z)}{f'(z)}=(\alpha-1)\frac{zg_1'(z)-1}{g_1(z)}+\cdots+(\alpha-1)\frac{zg_n'(z)-1}{g_n(z)}. \tag{14}$$

Then the expression

$$\left|\frac{zf''}{f'}\right| \tag{15}$$

can be evaluated as

$$\left|\frac{zf''(z)}{f'(z)}\right|=|\alpha-1|\left|\frac{zg_1'(z)-1}{g_1(z)}\right|+\cdots+|\alpha-1|\left|\frac{zg_n'(z)-1}{g_n(z)}\right|. \tag{16}$$

By multiplying the first and the last term of (16) with $\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}>0$, we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\le AB_1+\cdots+AB_n\le AC_1+\cdots+AC_n. \tag{17}$$

where

$$A=\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha-1|,$$

$$B_k = \left(\left| \frac{z g_k'(z)}{g_k(z)} \right| + 1 \right)$$

and

$$C_k = \left(\left| \frac{z^2 g_k'(z)}{g_k^2(z)} \right| \frac{|g_k(z)|}{|z|} + 1 \right) \forall k = \overline{1, n}.$$

By applying the Schwartz Lemma and using (17), we obtain

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq A D_1 + \cdots + A D_n, \tag{18}$$

where $D_k = \left(\left| \frac{z^2 g_k'(z)}{g_k^2(z)} - 1 \right| + 2 \right) \forall k = \overline{1, n}.$

Since $g_i \in T_2$, we have $\left| \frac{z^2 g_i'(z)}{g_i^2(z)} - 1 \right| < 1$, $\forall i = \overline{1, n}$. Further, from (18), we obtain:

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 3nA \leq \frac{3n |\alpha - 1|}{\operatorname{Re}\alpha}. \tag{19}$$

But $|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{3n}$ and from (19), we obtain that

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \tag{20}$$

for all $z \in U$. According to the Theorem 1 the function $F_{\alpha,n}$ is in the class S.

Theorem 4. *Let* $g_i \in T_{2,\mu}$, $g_i(z) = z + a_3^i z^3 + a_4^i z^4 + \ldots, \forall i = \overline{1, n}, n \in N^*, \alpha \in \mathbf{C}, \operatorname{Re}\alpha > 0$ *so that*

$$|\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{n(\mu + 2)}, \operatorname{Re}(n(\alpha - 1) + 1) \geq Re\alpha. \tag{21}$$

If $|g_i(z)| \leq 1, \forall z \in U, i = \overline{1, n}$ *then we have*

$$F_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z g_1^{\alpha-1}(t) \ldots g_n^{\alpha-1}(t)\, \mathrm{d}t \right)^{\frac{1}{n(\alpha-1)+1}} \in S. \quad (22)$$

Proof. Considering the same steps as in the above proof we obtain:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{\left(1-|z|^{2\mathrm{Re}\alpha}\right)}{\mathrm{Re}\alpha} |\alpha-1| \sum_{i=1}^{n} \left(\left| \frac{z^2 g_i'(z)}{g_i^2(z)} - 1 \right| + 2 \right). \quad (23)$$

But $f \in T_{2,\mu}$, which implies that $\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| < \mu, \forall z \in U$.
In these conditions we obtain:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq |\alpha-1| \frac{n(\mu+2)}{\mathrm{Re}\alpha}, \forall z \in U. \quad (24)$$

By applying the relation (21) we obtain that $\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \forall z \in U$.
So according to the Theorem 1 the function $F_{\alpha,\beta}$ is univalent.

Theorem 5. *Let* $g_i \in S(p), 0 < p < 2,\ g_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., \forall i = \overline{1,n}, n \in N^*, \alpha \in \mathbf{C}, \mathrm{Re}\alpha > 0$ *so that*

$$|\alpha-1| \leq \frac{\mathrm{Re}\alpha}{n(p+2)}, \mathrm{Re}(n(\alpha-1)+1) \geq Re\alpha. \quad (25)$$

If $|g_i(z)| \leq 1, \forall z \in U, i = \overline{1,n}$ *then we have*

$$F_{\alpha,n}(z) = \left((n(\alpha-1)+1) \int_0^z g_1^{\alpha-1}(t) \ldots g_n^{\alpha-1}(t)\, \mathrm{d}t \right)^{\frac{1}{n(\alpha-1)+1}} \in S. \quad (26)$$

Proof. Considering the same steps as in the above proof we obtain:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{\left(1-|z|^{2\mathrm{Re}\alpha}\right)|\alpha-1|}{|\alpha|\,\mathrm{Re}\alpha}\sum_{i=1}^{n}\left(\left|\frac{z^2 g_i'(z)}{g_i^2(z)}-1\right|+2\right). \tag{27}$$

But $g_i \in S(p), i=\overline{1,n}$ so

$$\left|\frac{z^2 g_i'(z)}{g_i^2(z)}-1\right| \le p|z|^2, \forall z \in U. \tag{28}$$

By applying (28) in (27), we obtain that:

$$\sum_{i=1}^{n}\left(\left|\frac{z^2 g_i'(z)}{g_i^2(z)}-1\right|+2\right) \le \sum_{i=1}^{n}\left(p|z|^2+2\right) \le \sum_{i=1}^{n}(p+2) = n(p+2). \tag{29}$$

In these conditions we obtain:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{|\alpha-1|\,n(p+2)}{\mathrm{Re}\alpha}, \forall z \in U. \tag{30}$$

By applying the relation (25) we obtain that $\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \le 1, \forall z \in U$. Thus, according to the Theorem 1, the function $F_{\alpha,n}$ is univalent.

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