THE NONCOMMUTATIVE DIRAC EQUATION FOR SOME WAVE FUNCTIONS

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Abstract. In this paper we will analyzed the noncommutative Dirac equation which was introduced in the paper [7], for some wave functions. For instance, we will study the movement of an electron inside of an atom using the noncommutative Dirac equation.

2005 Mathematics Subject Classification: 34L40, 58B99.

Keywords and phrases: noncommutative Dirac equation, wave functions.

1. Introduction

In the paper [7], we construct the "*noncommutative Dirac equation*", which was:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = Q(x,\partial_x)\psi(t,x) \tag{1}$$

Here $Q(x,\partial_x)$ represent the noncommutative harmonic oscilator which is the second-order ordinary differential operator:

$$Q(x,\partial_x) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\left(-\frac{\partial_x^2}{2}+\frac{x^2}{2}\right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(x\partial_x + \frac{1}{2}\right) =$$

$$\begin{pmatrix} -\frac{\alpha\partial_x^2}{2}+\alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix}$$

where α,β are two constants, $\alpha,\beta > 0$. $\psi(x,t)$ represent the wave function and $\hbar$ represent the Planck constant.

The wave function $\psi(x,t)$ describe the probability distribution in time and space of an particle.

For instance, the wave function $\psi(x,t)$ for an electron which is moving inside of an atom in the fundamental state, defined for $n = 1$ in the equation $E = -\frac{me^4}{8\varepsilon_0^2\hbar^2}\frac{1}{n^2}$, have the following wave function (see [4]): $\psi(x,t) = \sqrt{\frac{2}{\pi a^3}}e^{-\frac{r}{a}}\cos\omega t$ where $a = \frac{\hbar^2\varepsilon_0}{\pi me^2}$ and r represent the radius of the orbit of an electon.

2. Main Result

Theorem 2.1 *For the wave function* $\psi(x,t) = \sqrt{\frac{2}{\pi a^3}}e^{-\frac{r}{a}}\cos\omega t$ *, which represent the wave function of an electron which is moving inside of an atom, the noncommutative Dirac equation is:* $\hbar\omega\tan\omega t = x\partial_x + \frac{1}{2}$.

*Proof.*Starting with the noncommutative harmonic oscilator :

$$Q(x,\partial_x) = \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix}$$

and using the *noncommutative Dirac equation,* one obtain :

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix}\psi(t,x)$$

In this equation we take

$$\psi(t,x) = \sqrt{\frac{2}{\pi a^3}}e^{-\frac{r}{a}}\cos\omega t$$

and we get:

$$i\hbar\omega e^{-\frac{r}{a}}\sqrt{\frac{2}{\pi a^3}}(-\sin\omega t) = \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \frac{\alpha x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\frac{\beta\partial_x^2}{2} - \frac{\beta x^2}{2} \end{pmatrix}\sqrt{\frac{2}{\pi a^3}}e^{-\frac{r}{a}}\cos\omega t$$

Using the matricial representation for the complex numbers, $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, one obtain :

$$\begin{pmatrix} 0 & \hbar\frac{\partial\psi(t,x)}{\partial t} \\ -\hbar\frac{\partial\psi(t,x)}{\partial t} & 0 \end{pmatrix} = \begin{pmatrix} \left(-\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2}\right)\psi(t,x) & \left(-x\partial_x - \frac{1}{2}\right)\psi(t,x) \\ \left(x\partial_x + \frac{1}{2}\right)\psi(t,x) & \left(-\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2}\right)\psi(t,x) \end{pmatrix}$$

Identifying, one obtain the following equation:

$$\hbar\omega e^{-\frac{r}{a}}\sqrt{\frac{2}{\pi a^3}}\sin\omega t=\left(x\partial_x+\frac{1}{2}\right)\sqrt{\frac{2}{\pi a^3}}e^{-\frac{r}{a}}\cos\omega t$$

Finally, one obtain: $\hbar\omega\tan\omega t=-x\partial_x+\frac{1}{2}$, so the theorem is proved.

In the paper [7], we obtain the noncommutative Dirac equation for a free particle as:

$$\hbar\frac{\partial}{\partial t}\psi(x,t)=-\left(x\partial_x+\frac{1}{2}\right)\psi(x,t) \tag{2}$$

LEMMA 2.2. *If we take the deBroglie wave function* $\psi(t,x)=Ae^{\frac{iEt}{\hbar}}+Be^{-\frac{iEt}{\hbar}}$, *(where E represent the total energy), using the noncommutative Dirac equation (1), one obtain the total energy:*

$$E=-\frac{1}{i}\left(x\partial_x+\frac{1}{2}\right)\frac{Ae^{\frac{iEt}{\hbar}}+Be^{-\frac{iEt}{\hbar}}}{Ae^{\frac{iEt}{\hbar}}+Be^{-\frac{iEt}{\hbar}}}.$$

Proof. Applying equation (1) to the de Broglie wave function (2) , one obtain:

$$\hbar\left(A\frac{iE}{\hbar}e^{\frac{iEt}{\hbar}}-B\frac{iE}{\hbar}e^{-\frac{iEt}{\hbar}}\right)=-\left(x\partial_x+\frac{1}{2}\right)\left(Ae^{\frac{IEt}{\hbar}}+Be^{-\frac{iEt}{\hbar}}\right)\Rightarrow$$

iE$\left(Ae^{\frac{iEt}{\hbar}}-Be^{-\frac{IEt}{\hbar}}\right)=-\left(x\partial_x+\frac{1}{2}\right)\left(Ae^{\frac{iEt}{\hbar}}+Be^{-\frac{iEt}{\hbar}}\right)$

Finally, one obtain:

$$E=-\frac{1}{i}\left(x\partial_x+\frac{1}{2}\right)\frac{Ae^{\frac{iEt}{\hbar}}+Be^{-\frac{iEt}{\hbar}}}{Ae^{\frac{iEt}{\hbar}}-Be^{-\frac{iEt}{\hbar}}}$$

so, the lemma is proved.

COROLLARY 2.3. *If we consider a plane wave function:* $\psi(t,x)=ce^{\left(-\frac{i}{\hbar}Apx-\frac{iEBt}{\hbar}\right)}$, *where* A,B *are constants, using noncommutative Dirac equation one obtain the total energy:*

$$E=-\frac{1}{\hbar B}\left(px+\frac{\hbar}{2i}\right).$$

Proof.

$$\hbar\frac{\partial}{\partial t}\left(ce^{\left(\frac{ipAx}{\hbar}-\frac{iEBt}{\hbar}\right)}\right)=-\left(x\partial_x+\frac{1}{2}\right)ce^{\left(\frac{ipAx}{\hbar}-\frac{iEBt}{\hbar}\right)}\Rightarrow$$

$$\hbar ce^{\left(\frac{ipAx}{\hbar}-\frac{iEBt}{\hbar}\right)}\left(-\frac{iEB}{\hbar}\right)=-\left(x\partial_x+\frac{1}{2}\right)ce^{\left(\frac{ipAx}{\hbar}-\frac{iEBt}{\hbar}\right)}.$$

So, finally, we obtain:

$$iEB=x\partial_x+\frac{1}{2}\Rightarrow$$

$$E=\frac{1}{iB}\left(x\partial_x+\frac{1}{2}\right)=\frac{1}{\hbar B}\left(-\frac{i\hbar}{i^2}x\partial_x+\frac{\hbar}{2i}\right)=\frac{1}{\hbar B}(px+\frac{\hbar}{2i}).$$

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