SPLINE QUASI-INTERPOLANTS AND QUADRATURE FORMULAS

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Abstract. In this paper we study a new simple quadrature rule based on integrating a spline quasi-interpolant on a bounded interval. We also give error estimates for smooth functions.

2000 Mathematics Subject Classification: 65D30 , 65D32 , 41A55 , 41A15.

Keywords and phrases: quadrature rule, numerical integration, spline quasi-interpolant.

1. Introduction

Definition 1 [1] *The function $s(x)$ is called a spline function of degree d with knots $\{t_i\}_{i=1}^n$ if $-\infty := t_0 < t_1 < \cdots < t_n < t_{n+1} := \infty$ and*

i) for each $i = 0, \ldots, n$, $s(x)$ coincides on (t_i, t_{i+1}) with a polynomial of degree not greater then d;

ii) $s(x), s'(x), \ldots, s^{(d-1)}(x)$ are continuous functions on $(-\infty, +\infty)$.

We shall denote by $S_d(t_1, \ldots, t_n)$ the class of all spline functions of degree d with knots at $t_1, \ldots, t_n$. For fixed $\{t_i\}_{i=1}^n$, $S_d(t_1, \cdots, t_n)$ is a linear space and $\dim S_d(t_1, \cdots, t_n) = n + d + 1$.

Let $t_0 \le \cdots \le t_{d+1}$ be arbitrary points in $[a, b]$ such that $t_0 < t_{d+1}$.

Definition 2 [1] *The spline function*

$$B(t_0, \cdots, t_{d+1}; t) = (\cdot - t)_+^d \, [t_0, \cdots, t_{d+1}]$$

is called a B-spline of degree d with knots $t_0, \cdots, t_{d+1}$.

A property of B-spline it is:

$$\int_a^b B(t_0, \cdots, t_{d+1}; t)dt = \frac{1}{d+1}.$$

Given the sequence (finite or infinite) of points $\{t_i\}$, such that

$$\cdots \le t_i \le t_{i+1} \le \cdots$$

and $t_i < t_{i+d+1}$ for all i , we shall denote by $B_{i,d}(t)$ the B-spline

$$B_{i,d}(t) = (\cdot - t)_+^d [t_i, \cdots, t_{i+d+1}] .$$

The spline function $N_{i,d}(t) = (t_{i+d+1} - t_i)B_{i,d}(t)$ is called normalized B-spline .

THEOREM 1 [1] *Let* $a < t_{d+2} \le \cdots \le t_n < b$ *be fixed points such that* $t_i < t_{i+d+1}$ *for all admissible* i . *Choose arbitrary* $2d + 2$ *additional points* $t_1 \le \cdots t_{d+1} \le a$ *and* $b \le t_{n+1} \le \cdots \le t_{n+d+1}$ *and define* $B_i(t) = B(t_i, \cdots, t_{i+d+1}; t)$. *The B-spline* $B_1(t), \cdots, B_n(t)$ *constitute a basis for* $S_d(t_{d+2}, \cdots, t_n)$ *on* $[a, b]$.

In [3] is given a general construction of quasi-interpolants. Given a function f , the basic problem of spline approximation is to determine B-spline coefficients $(c_i)_{i=1}^n$ such that

$$Pf = \sum_{i=1}^{n} c_i N_{i,d}$$

is a reasonable approximation to f . The basic challenge is therefore to devise a procedure for determining the B-spline coefficients.

Let $t = (t_j)_{j=1}^{n+d+1}$ be arbitrary points in $[a, b]$, nondecreasing with $t_{d+1} = a$ and $t_{n+1} = b$. We assume that f is defined on $[a, b]$. We fix k and propose the following procedure for determining c_k:

1) Choose a local interval $I=(t_\mu,t_\nu)$ with the property that I intersects the support of $N_{k,d}$:

$$I\cap(t_k,t_{k+d+1})\neq\phi.$$

Denote the restriction of the space S_d to the interval I by $S_{d,I}$, namely

$$S_{d,I}=\text{span}\left\{N_{\mu-d,d},\cdots,N_{\nu-1,d}\right\}.$$

2) Choose some local approximation method P_I with the property that $P_Ig=g$ for all $g\in S_{d,I}$.

3) Let f_I denote the restriction of f to the interval I . Then there exist B-spline coefficients $(b_i)_{i=\mu-d}^{\nu-1}$ such that $P_If_I=\sum\limits_{i=\mu-d}^{\nu-1}b_iN_{i,d}$. Note that $\mu-d\leq k\leq\nu-1$ since supp $N_{k,d}$ intersects I .

4) Set $c_k=b_k$.

Theorem 2(de Boor-Fix) [3] *Let r be an integer with $0\leq r\leq d$ and let x_j be a number in $[t_j,t_{j+d+1}]$ for $j=1,\ldots,n$. Consider the quasi-interpolant*

$$Q_{d,r}f=\sum_{j=1}^{n}\lambda_j(f)N_{j,d} \tag{1}$$

where

$$\lambda_j(f)=\frac{1}{d}\sum_{k=0}^{r}(-1)^kD^{d-k}\rho_{j,d}(x_j)D^kf(x_j)$$

and $\rho_{j,d}(y)=(y-t_{j+1})\cdots(y-t_{j+d})$. Then $Q_{d,r}$ reproduces all polynomials of degree r and $Q_{d,d}$ reproduces all splines in S_d.

Suppose that $d\geq 2$ and fix an integer i such that $t_{i+d}>t_{i+1}$. We pick the largest subinterval $[a_i,b_i]=[t_l,t_{l+1}]$ of $[t_{i+1},t_{i+d}]$ and define the uniformly spaced points

$$x_{i,k}=a_i+\frac{k}{d}(b_i-a_i)\text{ for }k=0,1,\cdots,d \tag{2}$$

in this interval .

To define $P_d f \in S_d$ by

$$\begin{aligned} P_d f(x) &= \sum_{i=1}^{n} \lambda_i(f) N_{i,d}(x)\,, \quad \text{where} \\ \lambda_i(f) &= \sum_{k=0}^{d} w_{i,k} f(x_{i,k})\,. \end{aligned} \tag{3}$$

The following lemma show how the coefficients $(w_{i,k})_{k=0}^{d}$ should be chosen so that $P_d p = p$ for all $p \in \mathcal{P}_d$.

LEMMA 1 [3] *Suppose that in (3) the functionals* λ_i *are given by* $\lambda_i(f) = f(t_{i+1})$ *if* $t_{i+d} = t_{i+1}$ *, while if* $t_{i+d} > t_{i+1}$ *we set*

$$w_{i,k} = \gamma_i(p_{i,k})\,,\ k = 0, 1, \ldots, d$$

where $\gamma_i(p_{i,k})$ *is the i th B-spline coefficient of the polynomial*

$$p_{i,k}(x) = \prod_{j=0, j\neq k}^{d} \frac{x - x_{i,j}}{x_{i,k} - x_{i,j}}\,.$$

Then the operator P_d *in (3) satisfies* $P_d p = p$ *for all* $p \in \mathcal{P}_d$ *.*

LEMMA 2 [3] *Given a spline space* S_d *and numbers* $v_1, \ldots, v_d$ *. The i th B-spline coefficient of the polynomial* $p(x) = (x - v_1)\ldots(x - v_d)$ *can be written*

$$\gamma_i(p) = \frac{1}{d!} \sum_{(j_1,\ldots,j_d)\in\prod_d} (t_{i+j_1} - v_1)\ldots(t_{i+j_d} - v_d)\,,$$

where $\prod_d$ *is the set of all permutations of the integers* $1, 2, \ldots, d$*.*

Interesting results about spline quasi-interpolants were obtain by P. Sablonniere in [4], [5], [6], [7], [8].

We choose $-\infty \le a < b \le +\infty$ and let $w : (a, b) \to [0, +\infty\ )$ be a weight on the interval (a, b) . We denote

$$L_w^p = \{f : [a, b] \to \mathbb{R} \mid fw \text{ is measurable and } |f|^p \cdot w\,,\ p > 0 \text{ is integrable on } (a, b)\}\ .$$

Let

$$\begin{aligned}&\int_a^b f(t)w(t)dt = \sum_{k=1}^{n} c_{k,n} f(z_{k,n}) + \mathcal{R}_n[f] \\ &f \in L_w^1(a,b)\end{aligned} \tag{4}$$

be a quadrature formulae , where $z_{1,n} < z_{2,n} < \cdots < z_{n,n}$ are the points from $[a,b]$.

Definition 3 [2] *If the weight w is symmetric , namely $w(x) = w(a + b - x)$ any $x \in (a,b)$ and*

$$\begin{aligned}&c_{j,n} = c_{n+1-j,n} \\ &z_{j,n} = a + b - z_{n+1-j,n}\,,\ j = 1, 2, \ldots, n\end{aligned}$$

then (4) is called the symmetric quadrature formulae.

Theorem 3 [2] *If the quadrature formulae (4) is symmetric and*

$$\mathcal{R}_n[p] = 0\,,\ \ \textit{any } p \in \mathcal{P}_{2s}$$

then

$$\mathcal{R}_n[h] = 0\,,\ \ \text{any } h \in \mathcal{P}_{2s+1}\,.$$

Lemma 3 [2] *If $-\infty < \alpha < \beta < +\infty$ and w is a weight on (α,β) and*

$$\int_\alpha^\beta f(t)w(t)dt = \sum_{k=1}^{n} c_k f(z_k) + r_n[f]\,,\ f \in L_w^1(\alpha,\beta)$$

then

$$W(x) = w\left(\alpha + (\beta - \alpha)\frac{x-a}{b-a}\right)\,,\ x \in (a,b)\,,\ -\infty < a < b < +\infty$$

is a weight on (a,b) and

$$\int_a^b F(x)W(x)dx = \frac{b-a}{\beta-\alpha}\sum_{k=1}^{n} c_k F\left((a + (b-a)\frac{z_k-\alpha}{\beta-\alpha}\right) + \mathcal{R}_n[F]$$

where $F \in L^1_w(a,b)$ *and*

$$\mathcal{R}_n[F] = \frac{b-a}{\beta-\alpha} r_n[\tilde{F}]\,,\ \tilde{F}(t) = F\left(a + (b-a)\frac{t-\alpha}{\beta-\alpha}\right)\,.$$

We denote

$$W^r_p[a,b] := \left\{ f \in C^{r-1}[a,b]\,,\ f^{(r-1)} \text{ absolutely continuous }\,,\ \left\| f^{(r)} \right\|_p < \infty \right\}$$

with

$$\|f\|_p := \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}} \text{ for } 1 \le p < \infty$$
$$\|f\|_\infty := \text{supvrai}_{x\in[a,b]} |f(x)|\ .$$

Theorem 4 [1] *(Peano's theorem) Let $L(f)$ be an arbitrary linear functional defined in $W^r_1[a,b]$ such that the function $K(t) := L\left[(x-t)^{r-1}_+\right]$ is integrable over $[a,b]$. Suppose that $L(p) = 0$ for each polynomial $p \in \mathcal{P}_{r-1}$. Then*

$$L(f) = \frac{1}{(r-1)!} \int_a^b K(t) f^{(r)}(t) dt$$

for each $f \in W^r_1[a,b]$.

2. Main results

If in Lemma 1 we choose $d = 2$ and $t_1 = t_2 = t_3 = a$, $t_{n+1} = t_{n+2} = t_{n+3} = b$, we obtain the operator

$$P_2 f = \sum_{i=1}^{n} \lambda_i(f) N_{i,2} \ \text{ with}$$

$$\lambda_i(f) = -\frac{1}{2} f(t_{i+1}) + 2f\left(\frac{t_{i+1}+t_{i+2}}{2}\right) - \frac{1}{2} f(t_{i+2}) \tag{5}$$

which satisfy $P_2 p = p$ any $p \in \mathcal{P}_2$.

If we integrate the approximation formulae of function f

$$f(x)=\sum_{i=1}^{n}\lambda_i(f)N_{i,2}(x)+r_n[f]$$

to obtain following quadrature formulae with the exactness degree 2:

$$\int_a^b f(x)dx=\sum_{i=1}^{n}\frac{t_{i+3}-t_i}{3}\left[-\frac{1}{2}f(t_{i+1})+2f\left(\frac{t_{i+1}+t_{i+2}}{2}\right)-\frac{1}{2}f(t_{i+2})\right]+\mathcal{R}_n[f]. \tag{6}$$

For $n=3$ we have the Simpson′s quadrature formulae. We shall study the quadrature formulae for $n\geq 6$

We choose the equidistant nodes $(t_i)_{i=4}^{n}$ from the interval [a,b] and for simplicity of calculations we choose $a=0$, $b=1$. If denote $h=\dfrac{1}{n-2}$, we have $t_i=(i-3)h$, $i=\overline{4,n}$ and the quadrature formulae (6) can be written

$$\int_0^1 f(x)dx=h\left\{\frac{4}{3}f\left(\frac{h}{2}\right)-\frac{5}{6}f(h)+2\sum_{k=2}^{n-3}f\left(\frac{2k-1}{2}h\right)-\sum_{k=2}^{n-4}f(kh)-\right.$$
$$\left.\frac{5}{6}f\left((n-3)h\right)+\frac{4}{3}f\left(\frac{2n-5}{2}h\right)\right\}+\mathcal{R}_n[f]\,. \tag{7}$$

Because the quadrature formulae (7) is symmetric , from Theorem 3 following than the exactness degree of formulae (6) is equal with 3 and from Theorem 4, the remainder has the form

$$\mathcal{R}_n[f]=\frac{1}{6}\int_0^1 K(t)f^{(4)}(t)dt\ ,\ \text{where } f\in W_1^4[0,1]$$

$$K(t)=\mathcal{R}_n\left[(\cdot-t)_+^3\right]=\frac{(1-t)^4}{4}-h\left\{\frac{4}{3}\left(\frac{h}{2}-t\right)_+^3-\frac{5}{6}(h-t)_+^3+\right.$$
$$\left.2\sum_{k=2}^{n-3}\left(\frac{2k-1}{2}h-t\right)_+^3-\sum_{k=2}^{n-4}(kh-t)_+^3-\frac{5}{6}\left((n-3)h-t\right)_+^3+\frac{4}{3}\left(\frac{2n-5}{2}h-t\right)_+^3\right\}. \tag{8}$$

Lemma 4 *The Peano's kernel , definite in (8) verifies*

$$K(t) = K(1-t) \text{ any } t \in [0,1] \tag{9}$$

$$K(t) \geq 0 \text{ any } t \in [0,1] \tag{10}$$

$$\max_{t\in[0,1]} K(t) = \frac{h^4}{12} \tag{11}$$

$$\int_0^1 K(t)dt = \frac{1}{480} \cdot \frac{29n-88}{(n-2)^5} \,. \tag{12}$$

Proof. Using the symmetry of nodes and coefficients we obtain

$$K(1-t) = \frac{t^4}{4} - h\left\{\frac{4}{3}\left(t-\frac{h}{2}\right)_+^3 - \frac{5}{6}(t-h)_+^3 + 2\sum_{k=2}^{n-3}\left(t-\frac{2k-1}{2}h\right)_+^3 - \sum_{k=2}^{n-4}(t-kh)_+^3 - \frac{5}{6}(t-(n-3)h)_+^3 + \frac{4}{3}\left(t-\frac{2n-5}{2}h\right)_+^3\right\}. \tag{13}$$

If in the quadrature formulae (7) we choose $f(x) = (x-t)^3 \in \mathcal{P}_3$ we obtain

$$\frac{(1-t)^4}{4} - \frac{t^4}{4} = h\left\{\frac{4}{3}\left(\frac{h}{2}-t\right)^3 - \frac{5}{6}(h-t)^3 + 2\sum_{k=2}^{n-3}\left(\frac{2k-1}{2}h-t\right)^3 - \sum_{k=2}^{n-4}(kh-t)^3 - \frac{5}{6}((n-3)h-t)^3 + \frac{4}{3}\left(\frac{2n-5}{2}h-t\right)^3\right\}. \tag{14}$$

From the relations (8) , (13) , (14) and the formulae

$$(t_i-t)_+^3 - (t-t_i)_+^3 = (t_i-t)^3$$

we have $K(t) = K(1-t)$.

We denote $K(t) = K_j(t)$ for $t \in \left[\frac{(j-1)h}{2}, \frac{jh}{2}\right]$, $j = \overline{1, 2n-4}$.

From the relation (13) we obtain

$K_1(t)=\dfrac{t^4}{4};$

$K_2(t)=\dfrac{t^4}{4}-\dfrac{4}{3}h\left(t-\dfrac{h}{2}\right)^3;$

$K_3(t)=\dfrac{t^4}{4}-h\left\{\dfrac{4}{3}\left(t-\dfrac{h}{2}\right)^3-\dfrac{5}{6}\left(t-h\right)^3\right\};$

$K_{2i-1}(t)=\dfrac{t^4}{4}-h\left\{\dfrac{4}{3}\left(t-\dfrac{h}{2}\right)^3-\dfrac{5}{6}\left(t-h\right)^3+2\sum\limits_{k=2}^{i-1}\left(t-\dfrac{2k-1}{2}h\right)^3-\right.$
$\left.\sum\limits_{k=2}^{i-1}\left(t-kh\right)^3\right\}\ ,i=\overline{3,n-3};$

$K_{2i}(t)=\dfrac{t^4}{4}-h\left\{\dfrac{4}{3}\left(t-\dfrac{h}{2}\right)^3-\dfrac{5}{6}\left(t-h\right)^3+2\sum\limits_{k=2}^{i}\left(t-\dfrac{2k-1}{2}h\right)^3-\right.$
$\left.\sum\limits_{k=2}^{i-1}\left(t-kh\right)^3\right\}\ ,i=\overline{2,n-4};$

$K_{2i+1}(t)=\dfrac{t^4}{4}-h\left\{\dfrac{4}{3}\left(t-\dfrac{h}{2}\right)^3-\dfrac{5}{6}\left(t-h\right)^3+2\sum\limits_{k=2}^{i}\left(t-\dfrac{2k-1}{2}h\right)^3-\right.$
$\left.\sum\limits_{k=2}^{i}\left(t-kh\right)^3\right\}\ ,i=\overline{2,n-4}.$

From the relation (8) we have

$K_{2n-6}(t)=\dfrac{(1-t)^4}{4}-h\left\{-\dfrac{5}{6}\left((n-3)h-t\right)^3+\dfrac{4}{3}\left(\dfrac{2n-5}{2}h-t\right)^3\right\};$

$K_{2n-5}(t)=\dfrac{(1-t)^4}{4}-\dfrac{4}{3}h\left(\dfrac{2n-5}{2}h-t\right)^3;$

$K_{2n-4}(t)=\dfrac{(1-t)^4}{4}.$

We observe than

$$K_{2i}(t)=K_{2i-1}(t)-2h\left(t-\frac{2i-1}{2}h\right)^3\ ,\ i=\overline{2,n-4}; \tag{15}$$

$$K_{2i+1}(t)=K_{2i}(t)+h(t-ih)^3\ ,\ i=\overline{2,n-4}. \tag{16}$$

We have

$K_1'(t) = t^3 \geq 0$ for $t \in \left[0, \frac{h}{2}\right]$;

$K_2'(t) = (t-h)(t^2 - 3ht + h^2) \geq 0$ for $t \in \left[\frac{h}{2}, h\right]$;

$K_3'(t) = (t-h)\left(t - \frac{3h}{2}\right)(t+h) \leq 0$ for $t \in \left[h, \frac{3h}{2}\right]$;

$K_{2n-6}'(t) = \left(t - \frac{2n-7}{2}h\right)(t-(n-3)h)(t-(n-1)h) \geq 0$ for $t \in \left[\frac{(2n-7)h}{2}, (n-3)h\right]$;

$K_{2n-5}'(t) = (t-(n-3)h)\left[t^2 + (-2+3h)t + (h^2 - 3h + 1)\right] \leq 0$ for $t \in \left[(n-3)h, \frac{2n-5}{2}h\right]$;

$K_{2n-4}'(t) = -(1-t)^3 \leq 0$ for $t \in \left[\frac{2n-5}{2}h, 1\right]$.

We prove by induction

$$K_{2i}'(t) = \left(t - \frac{2i-1}{2}h\right)(t-ih)(t-(i+2)h) \quad , \; i = \overline{2, n-4}; \tag{17}$$

$$K_{2i+1}'(t) = (t-ih)\left(t - \frac{2i+1}{2}h\right)(t-(i-2)h) \quad , \; i = \overline{2, n-4}. \tag{18}$$

Now suppose than (17) and (18) hold for an arbitrary i . We have to prove than (17) and (18) hold for $i \to i+1$.

$$K_{2i+2}'(t) = K_{2i+1}'(t) - 6h\left(t - \frac{2i+1}{2}h\right)^2 =$$
$$(t-ih)\left(t - \frac{2i+1}{2}h\right)(t-(i-2)h) - 6h\left(t - \frac{2i+1}{2}h\right)^2 =$$
$$\left(t - \frac{2i+1}{2}h\right)(t-(i+1)h)(t-(i+3)h);$$
$$K_{2i+3}'(t) = K_{2i+2}'(t) + 3h(t-(i+1)h)^2 =$$
$$\left(t - \frac{2i+1}{2}h\right)(t-(i+1)h)(t-(i+3)h) + 3h(t-(i+1)h)^2 =$$
$$(t-(i+1)h)\left(t - \frac{2i+3}{2}h\right)(t-(i-1)h) .$$

We observe than

$$K'_{2i}(t) \geq 0 \text{ for } t \in \left[\frac{(2i-1)h}{2}, ih\right];$$

$$K'_{2i+1}(t) \leq 0 \text{ for } t \in \left[ih, \frac{(2i+1)h}{2}\right].$$

From elementary calculations we obtain

$$K_1(0) = 0\,,\ K_1\left(\frac{h}{2}\right) = \frac{h^4}{64}\,,\ K_2\left(\frac{h}{2}\right) = \frac{h^4}{64}\,,$$
$$K_2\left(h\right) = \frac{h^4}{12}\,,\ K_3\left(h\right) = \frac{h^4}{12}\,,\ K_3\left(\frac{3h}{2}\right) = \frac{7h^4}{192}\,,$$
$$K_{2i}\left(\frac{(2i-1)h}{2}\right) = \frac{7h^4}{192}\,,\ K_{2i}\left(ih\right) = \frac{h^4}{12}\,,\ \ i = \overline{2, n-4}\,,$$
$$K_{2i+1}\left(ih\right) = \frac{h^4}{12}\,,\ K_{2i+1}\left(\frac{(2i+1)h}{2}\right) = \frac{7h^4}{192}\,,\ \ i = \overline{2, n-4}\,,$$
$$K_{2n-6}\left(\frac{(2n-7)h}{2}\right) = \frac{7h^4}{192}\,,\ K_{2n-6}\left((n-3)h\right) = \frac{h^4}{12}\,,\ K_{2n-5}\left((n-3)h\right) = \frac{h^4}{12}\,,$$
$$K_{2n-5}\left(\frac{(2n-5)h}{2}\right) = \frac{h^4}{64}\,,\ K_{2n-4}\left(\frac{(2n-5)h}{2}\right) = \frac{h^4}{64}\,,\ K_{2n-4}(1) = 0\,.$$

Therefore $K(t) \geq 0$ any $t \in [0,1]$ and $\max\limits_{t\in[0,1]} K(t) = \dfrac{h^4}{12}$.

$$\int_0^1 K(t)dt = \frac{1}{20} - h^5\left\{\frac{-9}{48} + \frac{1}{32}\sum_{k=2}^{n-3}(2k-1)^4 - \frac{1}{4}\sum_{k=2}^{n-4}k^4 - \frac{5}{24}(n-3)^4 + \right.$$
$$\left.\frac{(2n-5)^4}{48}\right\} = \frac{1}{480}\cdot\frac{29n-88}{(n-2)^5}\,.$$

Theorem 5 *If $f \in W_1^4[0,1]$, $n \geq 6$ and there exist real numbers m, M such that $m \leq f^{(4)}(t) \leq M$, $t \in [0,1]$, then*

$$|\mathcal{R}[f]| \leq \frac{29n-88}{2880(n-2)^5}\left\{\frac{M-m}{2} + \left|\frac{M+m}{2}\right|\right\}. \tag{19}$$

Proof. We can write

$$\mathcal{R}[f] = \frac{1}{6}\int_0^1 K(t) f^{(4)}(t)dt =$$

$$\frac{1}{6}\int_0^1 K(t)\left[f^{(4)}(t) - \frac{m+M}{2}\right]dt + \frac{1}{6}\cdot\frac{m+M}{2}\cdot\int_0^1 K(t)dt =$$

$$\frac{1}{6}\int_0^1 K(t)\left[f^{(4)}(t) - \frac{m+M}{2}\right]dt + \frac{m+M}{2}\cdot\frac{1}{2880}\cdot\frac{29n-88}{(n-2)^5}.$$

Therefore

$$|\mathcal{R}[f]| \leq \frac{1}{6}\max_{t\in[0,1]}\left|f^{(4)}(t) - \frac{m+M}{2}\right|\cdot\int_0^1 |K(t)|\,dt + \left|\frac{m+M}{2}\right|\cdot\frac{29n-88}{2880(n-2)^5} = \tag{20}$$

$$\frac{29n-88}{2880(n-2)^5}\left\{\max_{t\in[0,1]}\left|f^{(4)}(t) - \frac{m+M}{2}\right| + \left|\frac{m+M}{2}\right|\right\}.$$

But $m \leq f^{(4)}(t) \leq M$, that is

$$\left|f^{(4)}(t) - \frac{m+M}{2}\right| \leq \frac{M-m}{2} \tag{21}$$

From (20) and (21) we have

$$|\mathcal{R}[f]| \leq \frac{29n-88}{2880(n-2)^5}\left\{\frac{M-m}{2} + \left|\frac{M+m}{2}\right|\right\}.$$

THEOREM 6 *Let $f \in W_1^4[0,1]$, $n \geq 6$. If there exist a real number m such that $m \leq f^{(3)}(t)$, $t \in [0,1]$, then*

$$|\mathcal{R}[f]| \leq \frac{1}{72(n-2)^4}\cdot\left[T - m + \frac{29n-88}{40(n-2)}|m|\right] \tag{22}$$

where $T = f^{(3)}(1) - f^{(3)}(0)$.

If there exist a real number M such that $f^{(3)}(t) \leq M$, $t \in [0,1]$, then

$$|\mathcal{R}[f]| \leq \frac{1}{72(n-2)^4}\cdot\left[M - T + \frac{29n-88}{40(n-2)}|M|\right]. \tag{23}$$

Proof. We can write

$$\mathcal{R}[f]=\frac{1}{6}\int_0^1 K(t)f^{(4)}(t)dt=\frac{1}{6}\int_0^1 K(t)\left(f^{(4)}(t)-m\right)dt+\frac{m}{6}\int_0^1 K(t)dt=$$

$$\frac{1}{6}\int_0^1 K(t)\left(f^{(4)}(t)-m\right)dt+m\cdot\frac{29n-88}{2880(n-2)^5}\,.$$

Therefore

$$|\mathcal{R}[f]|\leq\frac{1}{6}\max_{t\in[0,1]}|K(t)|\int_0^1\left(f^{(4)}(t)-m\right)dt+|m|\cdot\frac{29n-88}{2880(n-2)^5}=$$

$$\frac{1}{72(n-2)^4}\left[f^{(3)}(1)-f^{(3)}(0)-m\right]+|m|\cdot\frac{29n-88}{2880(n-2)^5}=$$

$$\frac{1}{72(n-2)^4}\left[T-m+\frac{29n-88}{40(n-2)}|m|\right]\,.$$

In a similar way we can prove than (23) holds.

Using Lemma 3 and Lemma 4 and denoting $\tilde{h}=\dfrac{b-a}{n-2}$, for $f\in C^4[a,b]$, the quadrature formulae (7) can be written

$$\int_a^b f(x)dx=\frac{(b-a)}{n-2}\left\{\frac{4}{3}f\left(a+\frac{\tilde{h}}{2}\right)-\frac{5}{6}f\left(a+\tilde{h}\right)+2\sum_{k=2}^{n-3}f\left(a+\frac{2k-1}{2}\tilde{h}\right)-\right.$$
$$\left.\sum_{k=2}^{n-4}f\left(a+k\tilde{h}\right)-\frac{5}{6}f\left(a+(n-3)\tilde{h}\right)+\frac{4}{3}f\left(a+\frac{2n-5}{2}\tilde{h}\right)\right\}+$$
$$\frac{29n-88}{2880(n-2)^5}(b-a)^5f^{(4)}(\xi)\ ,\ a\leq\xi\leq b\,. \tag{24}$$

If we integrate the approximation formulae of function f , obtained from (1) , and we choose $d=r=2$ then we obtain following the quadrature formulae with the exactness degree 2 :

$$\int_a^b f(x)dx=\sum_{j=1}^{n}\frac{t_{j+3}-t_j}{3}\left\{f(x_j)-\frac{1}{2}\left[2x_j-(t_{j+1}+t_{j+2})\right]f'(x_j)+\right.$$
$$\left.\frac{1}{2}(x_j-t_{j+1})(x_j-t_{j+2})f''(x_j)\right\} \tag{25}$$

We choose $x_j = \dfrac{t_{j+1}+t_{j+2}}{2}$ and the equidistant nodes $(t_i)_{i=4}^{n}$ from the interval [a,b] and for simplicity of calculations we choose $a = 0$, $b = 1$. If denote $h = \dfrac{1}{n-2}$, we have $t_i = (i-3)h$, $i = \overline{4,n}$ and the quadrature formulae (25) can be written

$$\int_0^1 f(x)dx = h\left\{\frac{1}{3}f(0) + \frac{2}{3}f\left(\frac{h}{2}\right) + \sum_{k=2}^{n-3} f\left(\frac{2k-1}{2}h\right) + \frac{2}{3}f\left(\frac{2n-5}{2}h\right) + \frac{1}{3}f(1) - \frac{h^2}{8}\left[\frac{2}{3}f''\left(\frac{h}{2}\right) + \sum_{k=2}^{n-3} f''\left(\frac{2k-1}{2}h\right) + \frac{2}{3}f''\left(\frac{2n-5}{2}h\right)\right]\right\} + \mathcal{R}_n[f] \,. \tag{26}$$

The exactness degree of formulae (26) is 3 , and the remainder has the form

$$\mathcal{R}_n[f] = \frac{1}{6}\int_0^1 K(t)f^{(4)}(t)dt \ , \ \text{ where } \ f \in W_1^4[0,1]$$

$$K(t) = \mathcal{R}_n\left[(\cdot - t)_+^3\right] = \frac{(1-t)^4}{4} - h\left\{\frac{2}{3}\left(\frac{h}{2}-t\right)_+^3 + \sum_{k=2}^{n-3}\left(\frac{2k-1}{2}h - t\right)_+^3 + \frac{2}{3}\left(\frac{2n-5}{2}h - t\right)_+^3 + \frac{1}{3}(1-t)^3 - \frac{3h^2}{4}\left[\frac{2}{3}\left(\frac{h}{2}-t\right)_+ + \sum_{k=2}^{n-3}\left(\frac{2k-1}{2}h - t\right)_+ + \frac{2}{3}\left(\frac{2n-5}{2}h - t\right)_+\right]\right\} . \tag{27}$$

Lemma 5. *The Peano's kernel, definite in (27) verifies*

$$K(t) = K(1-t) \ \text{ any } \ t \in [0,1] \tag{28}$$

$$-\frac{5}{192}h^4 \le K(t) \le \frac{h^4}{12} \ \text{ any } \ t \in [0,1] \tag{29}$$

$$\int_0^1 K(t)dt = \frac{1}{960}\cdot\frac{43n-136}{(n-2)^5} \,. \tag{30}$$

Proof. To prove in a similar way than Lemma 4.

Using Lemma 3 and denoting $\tilde{h} = \dfrac{b-a}{n-2}$, the quadrature formulae (26) can be written

$$\int_a^b f(x)dx = \tilde{h}\left\{\frac{1}{3}f(a) + \frac{2}{3}f\left(a+\frac{\tilde{h}}{2}\right) + \sum_{k=2}^{n-3} f\left(a+\frac{2k-1}{2}\tilde{h}\right) + \frac{2}{3}f\left(a+\frac{2n-5}{2}\tilde{h}\right) + \frac{1}{3}f(b) - \frac{\tilde{h}^2}{8}\left[\frac{2}{3}f''\left(a+\frac{\tilde{h}}{2}\right) + \sum_{k=2}^{n-3} f''\left(a+\frac{2k-1}{2}\tilde{h}\right) + \frac{2}{3}f''\left(a+\frac{2n-5}{2}\tilde{h}\right)\right]\right\} + \tilde{\mathcal{R}}_n[f]. \tag{31}$$

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