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NEW GENERALIZATIONS OF AHLFOR'S, BECKER'S AND PASCU'S UNIVALENCE CRITERIONS

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Abstract. In this paper we obtain new generalizations of Ahlfors's, Becker's and Pascu's univalence criterions in the open unit disk for the integral

$$\left[\alpha \int_0^z u^{\alpha-1} f'(u) du\right]^{\frac{1}{\alpha}},$$

the function $f \in \mathcal{A}$ and α complex number, $\alpha \neq 0$.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be the subclass of univalent functions in the class $\mathcal{A}$.

We will use the following lemmas for proving our main results.

Lemma 1. *(Pescar [7]). Let α and c be complex numbers, Re $\alpha > 0$, $|c| \leq 1$, $c \neq -1$ and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$.*

If

$$\left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \tag{1}$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du\right]^{\frac{1}{\alpha}} \tag{2}$$

is regular and univalent in $\mathcal{U}$.

Lemma 2. *(Schwarz [4]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \ (z \in \mathcal{U}_R), \tag{3}$$

the equality (in the inequality (3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

2. Main results

Theorem 1. *Let α and c be complex numbers, $Re\ \alpha > 0$, $|c| \leq 1$, $c \neq -1$ and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \dots$.*

If

$$|c||z|^{2Re\ \alpha} + \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{4}$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1}f'(u)du\right]^{\frac{1}{\alpha}} \tag{5}$$

is regular and univalent in $\mathcal{U}$.

Proof. For $z = 0$, the condition (4) is verified.

If $z \in \mathcal{U}$, $z \neq 0$, then we have

$$\left|\frac{1-|z|^{2\alpha}}{\alpha}\right| = \left|\frac{1-e^{2\alpha\ln|z|}}{\alpha}\right| = \left|2\ln|z|\int_0^1 e^{2\alpha t\ln|z|}dt\right| \leq$$
$$\leq -2\ln|z|\int_0^1\left|e^{2\alpha t\ln|z|}\right|dt = -2\ln|z|\int_0^1 e^{2tRe\ \alpha\cdot\ln|z|}dt =$$
$$= \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha},$$

for $Re\ \alpha > 0$ and, hence we have

$$\left|\frac{1-|z|^{2\alpha}}{\alpha}\right| \leq \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}, \tag{6}$$

for all $z \in \mathcal{U}$. We consider the function $w(z) = c|z|^{2\alpha} + \frac{1-|z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)}$, $z \in \mathcal{U}$. Using the function $w(z)$ and the relation (6) we obtain that

$$\left| c|z|^{2\alpha} + \frac{1-|z|^{2\alpha}}{\alpha} \cdot \frac{zf''(z)}{f'(z)} \right| \leq |c| \left| |z|^{2\alpha} \right| + \left| \frac{1-|z|^{2\alpha}}{\alpha} \right| \left| \frac{zf''(z)}{f'(z)} \right|$$
$$\leq |c| \, |z|^{2Re\,\alpha} + \frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha} \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \mathcal{U},$$

hence, by using the hypothesis (4), we obtain

$$\left| c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \tag{7}$$

for all $z \in \mathcal{U}$.

From (7) and Lemma 1 it results that the function $F_\alpha(z)$ defined in (5) is regular and univalent in $\mathcal{U}$. □

Using Theorem 1 we obtain new results for the univalence of integral operator F_α defined by (5).

Theorem 2. *Let α and c be complex numbers, $Re\,\alpha > 0$, $|c| \leq 1$, $c \neq -1$ and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$.*

If

$$|c||z|^{2Re\,\alpha} + \frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{8}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\,\beta \geq Re\,\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{\frac{1}{\beta}} \tag{9}$$

is in the class $\mathcal{S}$.

Proof. We consider the function $\psi(x) = \frac{1-a^{2x}}{x}$, $x \in (0, \infty)$, $0 < a < 1$. The function $\psi(x)$ is the function decreasing for $x \in (0,1)$. If $x_1 = Re\,\alpha \leq x_2 = Re\,\beta$ and $a = |z|$, $z \in \mathcal{U}$, then

$$\frac{1-|z|^{2Re\,\beta}}{Re\,\beta} \leq \frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}, \tag{10}$$

for all $z \in \mathcal{U}$. From (10) we obtain

$$\frac{1-|z|^{2Re\,\beta}}{Re\,\beta} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha} \left| \frac{zf''(z)}{f'(z)} \right|, \tag{11}$$

for all $z \in \mathcal{U}$.

Let's consider the function $\varphi(x) = a^x$, $x \in (0, \infty)$, $0 < a < 1$. For $x_1 = Re\ \alpha \leq x_2 = Re\ \beta$ and $a = |z|$, $z \in \mathcal{U}$, then

$$|z|^{2Re\ \beta} \leq |z|^{2Re\ \alpha}, \quad (z \in \mathcal{U}). \tag{12}$$

From (11) and (12) we get

$$|c|\,|z|^{2Re\ \beta} + \frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq |c|\,|z|^{2Re\ \alpha} + \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \tag{13}$$

for all $z \in \mathcal{U}$, and hence, by (8) we obtain

$$|c|\,|z|^{2Re\ \beta} + \frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{14}$$

for all $z \in \mathcal{U}$.

From (14) and by Theorem 1 we obtain that the function $F_\beta(z)$ belongs to the class $\mathcal{S}$. □

Theorem 3. *Let α and c be complex numbers, $Re\ \alpha > 0$, $|c| < 1$ and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$.*

If

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}(1-|c|), \tag{15}$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u)du\right]^{\frac{1}{\alpha}} \tag{16}$$

is regular and univalent in $\mathcal{U}$.

Proof. Let's consider the function $p(z) = \frac{zf''(z)}{f'(z)}$, $z \in \mathcal{U}$. We have $p(0) = 0$ and from (15) by Lemma 2, we get

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}(1-|c|)|z|, \quad (z \in \mathcal{U}). \tag{17}$$

From (17) we obtain

$$|c|\,|z|^{2Re\ \alpha} + \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq$$

$$\leq |c|\,|z|^{2Re\ \alpha} + \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}(1-|c|), \tag{18}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\leq 1}\left[\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}|z|\right] = \frac{2}{(2Re\,\alpha+1)^{\frac{2Re\,\alpha+1}{2Re\,\alpha}}},$$

from (18) we get

$$|c|\,|z|^{2Re\,\alpha} + \frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad (z \in \mathcal{U}), \tag{19}$$

and hence, by Theorem 1, it results that the integral operator F_α is in the class $\mathcal{S}$. □

Using these results we obtain the remarks.

Remark 1. *If $\alpha = 1$, from Theorem 1, we obtain Ahlfors's and Becker's criterion of univalence [1], [3].*

Remark 2. *For $c = 0$ and $\alpha = 1$, from Theorem 1, we obtain Becker's criterion of univalence [2].*

Remark 3. *For $c = 0$, from Theorem 1, we obtain the criterion of univalence proved in [5].*

Remark 4. *For $c = 0$, from Theorem 2, we obtain the criterion of univalence proved in [6].*

Remark 5. *Theorem 1 is an improvement of univalence criterion prooved in Lemma 1 [7].*

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