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INCOMPLETE MITTAG-LEFFLER FUNCTION

Dharmendra Kumar Singh and Saurabh Porwal

Abstract. This paper devoted to the study of incomplete Mittag-Leffler function and some of its properties in terms of incomplete Wright function.

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1. Introduction

In the mathematical literature, it is well-known the importance that have the Mittag Leffler function due to Swedish mathematician Gosta Mittag-Leffrer [7] in 1903

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1}$$

where z is a complex variable is called the Mittag-Leffler function of order α. The Mittag Leffler function is a direct generalization of the exponnential function $e^z=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)}$ and admits a first generalization given by two parameter Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta\in C;\Re(\alpha)>0,\Re(\beta)>0), \tag{2}$$

which is known as Wiman's function or generalized Mittag-Leffler [3] function as $E_{\alpha,1}(z)=E_{\alpha}(z)$.

In 1971, Prabhakar [10] introduced the Mittag-Leffler type function $E^{\gamma}_{\alpha,\beta}(z)$ defined by

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{3}$$

where α,β and γ are complex number; $\Re(\alpha)>0$, $\Re(\beta)>0$ and $(\gamma)_n$ the Pochhammer symbol [6] given by

$$(\gamma)_n=\gamma(\gamma+1)...(\gamma+n-1)=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},$$

where $\Gamma(z)$ is the classical Gamma function (see [6]) defined by the following integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re(z) > 0. \tag{4}$$

In 2007 Shukla and Prajapati [2] introduced and investigated a further generalization of Mittag-Leffler function. $E_{\alpha,\beta}^{\gamma,q}(z)$ of which is defined for $\alpha, \beta, \gamma \in C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $q \in (0,1) \cup N$ in the following way

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{5}$$

where $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol (Rainville [6]) which in particular reduces to $q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n$ if $q \in N$.

In this paper, we introduce incomplete Mittag-Leffler function with the help of incomplete Pochhammer symbol in the following way

$$E_{\alpha,\beta}^{[\delta,x]}(z) = \sum_{k=0}^{\infty} \frac{[\delta, x]_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \tag{6}$$

and

$$E_{\alpha,\beta}^{(\delta,x)}(z) = \sum_{k=0}^{\infty} \frac{(\delta, x)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \tag{7}$$

where $\alpha, \beta, \delta \in C; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0$ and $[\delta, x]_k$ and $(\delta, x)_k$ represent incomplete Pocchammer symbol which is introduced by Srivastava et al. [8] and defined as follows

$$(\lambda; x)_\nu = \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)}, \quad (\lambda, \nu \in C; x \geq 0) \tag{8}$$

$$[\lambda; x]_\nu = \frac{\Gamma(\lambda + \nu, x)}{\Gamma(\lambda)}, \quad (\lambda, \nu \in C; x \geq 0) \tag{9}$$

and these incomplete Pochhammer symbol satisfy the following decomposition relation

$$(\lambda; x)_\nu + [\lambda; x]_\nu = (\lambda)_\nu, \quad (\lambda, \nu \in C; x \geq 0), \tag{10}$$

where incomplete gamma function $\gamma(s, x)$ and $\Gamma(s, x)$ defined by

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt \quad (\Re(s) > 0; x \geq 0) \tag{11}$$

and

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt \quad (\Re(s) > 0; x \geq 0) \tag{12}$$

and satisfy the following decomposition formula

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s), \quad (\Re(s) > 0). \tag{13}$$

2. $E^{[\delta,x]}_{\alpha,\beta}(z)$ and $E^{(\delta,x)}_{\alpha,\beta}(z)$ in terms of Incomplete generalized hypergeometric function

If A_p the array of p parameters like as $a_1, a_2, ..., a_p$. Then the Pochhammer symbol $(A_p)_n$, and the incomplete Pochhammer symbols $(A_p; x)_n$ and $[A_p; x]_n$ are defined by

$$(A_p)_n := (a_1)_n (a_2)_n ... (a_p)_n. \tag{14}$$

$$(A_p; x)_n := (a_1; x)_n (a_2)_n ... (a_p)_n. \tag{15}$$

$$[A_p; x]_n := [a_1; x]_n (a_2)_n ... (a_p)_n, \tag{16}$$

and its decomposition formula is defined as

$$\begin{aligned} (A_p; x)_n + [A_p; x]_n &= [(a_1; x)_n + [a_1; x]_n](a_2)_n ... (a_p)_n \\ &= (a_1)_n (a_2)_n ... (a_p)_n \\ &= (A_p)_n \end{aligned}$$

then by putting $\delta = A_p$ and $\alpha = n$ in (6) and (7) and using the formula

$$(\lambda)_{mn} = m^{mn} \Pi_{j=1}^{m} \left(\frac{\lambda + j - 1}{m} \right)_n (n \in N_0; m \in N)$$

we obtain incomplete generalized hypergeometric function form which is recently introduced by Srivastava et al. [8]

$$\begin{aligned} E^{[A_p,x]}_{n,\beta}(z) &= \sum_{k=0}^{\infty} \frac{[A_p, x]_k}{\Gamma(nk+\beta)} \frac{z^k}{k!} \\ &= \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{[a_1, x]_k (a_2)_x ... (a_p)_x}{\prod_{j=1}^{n} \left(\frac{\beta+j-1}{n}\right)_k} \frac{\left(\frac{z}{n^n}\right)^k}{k!} \\ &= \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{[a_1, x]_k (a_2)_x ... (a_p)_x}{\left(\frac{\beta}{n}\right)_k \left(\frac{\beta+1}{n}\right)_k ... \left(\frac{\beta+n-1}{n}\right)_k} \frac{\left(\frac{z}{n^n}\right)^k}{k!} \\ &= \frac{1}{\Gamma(\beta)} {}_p\Gamma_n \left[\begin{array}{c} (a_1, x), a_2, ..., a_p \\ \frac{\beta}{n}, \frac{\beta+1}{n}, ..., \frac{\beta+n-1}{n} \end{array} \middle| \frac{z}{n^n} \right] \end{aligned} \tag{17}$$

and

$$E_{n,\beta}^{(A_p,x)}(z) = \frac{1}{\Gamma(\beta)} {}_p\gamma_n \left[\begin{array}{c} (a_1, x)\,,\, a_2, ..., a_p \\ \frac{\beta}{n}, \frac{\beta+1}{n}, ..., \frac{\beta+n-1}{n} \end{array} \middle| \frac{z}{n^n} \right], \tag{18}$$

where

$$|(\lambda; x)_n| \leq |(\lambda)_n| \text{ and } |[\lambda; x]_n| \leq |(\lambda)_n| \tag{19}$$

$(n \in N_0; \lambda \in C; x > 0)$, which is the sufficient condition that the infinite series would converge absolutely can be derived from those that are well-documated in the case of the generalized hypergeometric function ${}_pF_q(p, q \in N_0)$.

3. Incomplete Wright Function

The generalized Wright function ${}_p\Psi_q(z)$ defined for $z \in C$, $a_j, b_j \in C$ and $\alpha_i, \beta_j \in \Re$ $(\alpha_i, \beta_j \neq 0; i = 0, 1, 2, ..., p,\ j = 1, 2, 3, .., q)$ is given by the series

$${}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{array}{c} (a_i, \alpha_i)_{(1,p)} \\ (b_j, \beta_j)_{(1,q)} \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i s) z^k}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j s) k!}, \tag{20}$$

where C is the set of the complex number and $\Gamma(z)$ is the Euler gamma function[1, section 1.1] and the function (20) introduced by Wright [5] and known as generalized Wright function.Condition for the existence of the generalized Wright in terms of H-function were established in [4]. The particular ${}_p\Psi_q(z)$ is an entire function if there holds the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 1. \tag{21}$$

Now with the help of incomplete Pochhammer symbol (8) and (9) we introduce the incomplete generalized hypergeometric Wright function

$${}_p\overline{\Psi}_q(z) = {}_p\overline{\Psi}_q \left[\begin{array}{c} [a_1, \alpha_1, x] \ldots (a_p, \alpha_p) \\ (b_1, \beta_1, x) \ \ldots (b_q, \beta_q) \end{array} \middle| z \right]$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + \alpha, k, x)\Gamma(a_2 + \alpha_2 k)...\Gamma(a_p + \alpha_p k)}{\Gamma(b_1 + \beta_1 k, x)\Gamma(b_2 + \beta_2 k)...\Gamma(b_q + \beta_q k)} \frac{z^k}{k!}, \tag{22}$$

and

$${}_p\underline{\Psi}q(z) = {}_p\underline{\Psi}q \left[\begin{array}{c} (a_1, \alpha_1, x) \ldots (a_p, \alpha_p) \\ (b_1, \beta_1, x) \ldots (b_q, \beta_q) \end{array} \middle| z \right]$$

$$= \sum_{k=0}^{\infty} \frac{\gamma(a_1 + \alpha_1 k, x)\Gamma(a_2 + \alpha_2 k)...\Gamma(a_p + \alpha_p k)}{\gamma(b_1 + \beta_1 k, x)\Gamma(b_2 + \beta_2 k)...\Gamma(b_q + \beta_q k)} \frac{z^k}{k!}, \tag{23}$$

provided that the defining the infinite series in each case is absolutely convergent if satisfy the condition (19). By decomposition formula (10), equation (22) and (23) can be written in terms of generalized hypergeometric Wright function (20).

It should be worthy to note that if we take $x = 0$ in equation (22) and (23) then its reduce immediately in (20).

4. Incomplete Mittag-Leffler function in terms of Incomplete Wright function

In this section we write incomplete Mittag-leffler function (6) and (7) function in terms of incomplete Wright generalized function (22) and (23)

$$E_{\alpha,\beta}^{[\delta,x]} = \sum_{k=0}^{\infty} \frac{[\delta, x]_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}$$

$$= \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{\Gamma(\delta+k, x)}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}$$

$$= \frac{1}{\Gamma(\delta)} {}_1\overline{\Psi}_1 \left[\begin{matrix} [\delta, 1, x] \\ (\beta, \alpha) \end{matrix} | z \right] \tag{24}$$

and

$$E_{\alpha,\beta}^{(\delta,x)} = \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{\gamma(\delta+k, x)}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}$$

$$= \frac{1}{\Gamma(\delta)} {}_1\underline{\Psi}_1 \left[\begin{matrix} (\delta, 1, x) \\ (\beta, \alpha) \end{matrix} | z \right], \tag{25}$$

where $\alpha, \beta, \delta \in C; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0$.
In the further section we will be use only equation (6), because formula (10) show that it is sufficient to discuss the properties and characteristics of the incomplete Mittag-Leffler function.

5. Laplace transform of incomplete Mittag-Leffler function

The Laplace transform of the function $f(z)$ is defined as [9]

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt, \qquad \Re(s) > 0. \tag{26}$$

Theorem 5.1. *If* $\alpha, \beta, \delta, \sigma \in C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0, \Re(p) > 0, \Re(s) > 0,$ $\Re(\sigma) > 0$ *then*

$$\int_0^\infty z^{p-1} e^{-sz} E_{\alpha,\beta}^{[\delta,x]}(xz^\sigma)dz = \frac{s^{-p}}{\Gamma(\delta)} {}_2\overline{\Psi}_1 \left[\begin{array}{l} [\delta, 1, x]\,(p, \sigma) \,|\, \frac{x}{s^\sigma} \\ (\beta, \alpha) \end{array} \right]. \tag{27}$$

Proof.

$$\int_0^\infty z^{p-1} e^{-sz} E_{\alpha,\beta}^{[\delta,x]}(xz^\sigma)dz = \int_0^\infty z^{p-1} e^{-sz} \sum_{k=0}^\infty \frac{[\delta, x]_k}{\Gamma(\alpha k + \beta)} \frac{(xz^\sigma)^k}{k!} dz$$

$$= \sum_{k=0}^\infty \frac{[\delta, x]_k}{\Gamma(\alpha k + \beta)} \frac{x^k}{k!} \int_0^\infty e^{-sz} z^{\sigma k + p - 1} dz$$

$$= \frac{s^{-p}}{\Gamma(\delta)} \sum_{k=0}^\infty \frac{\Gamma(\delta + k, x)\,\Gamma(\sigma k + p)}{\Gamma(\alpha k + \beta)\, k!} \left[\frac{x}{s^\sigma} \right]^k$$

$$= \frac{s^{-p}}{\Gamma(\delta)} {}_2\overline{\Psi}_1 \left[\begin{array}{l} [\delta, 1, x]\,(p, \sigma) \,|\, \frac{x}{s^\sigma} \\ (\beta, \alpha) \end{array} \right]. \tag{28}$$

6. Euler Beta Transforms of incomplete Mittag-Leffler function

The Beta transform of a function $f(z)$ defined as (see [9])

$$B(f(z); a, b) = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz, \;\; \Re(a) > 0, \Re(b) > 0. \tag{29}$$

Theorem 6.1 *If* $a, b, \alpha, \beta, \delta, \sigma \in C,$ $\Re(a) > 0,$ $\Re(b) > 0,$ $\Re(\alpha) > 0,$ $\Re(\beta) > 0,$ $\Re(\delta) > 0,$ $\Re(\sigma) > 0,$ *then*

$$\int_0^1 z^{a-1}(1-z)^{b-1} E_{\alpha,\beta}^{[\delta,x]}(xz^\sigma)dz = \frac{\Gamma(b)}{\Gamma(\delta)} {}_2\overline{\Psi}_2 \left[\begin{array}{ll} [\delta, 1, x] & (a, \sigma) \\ (\beta, \alpha) & (a + b, \sigma) \end{array} \,|\, x \right]. \tag{30}$$

Proof.

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{\alpha,\beta}^{[\delta,x]}(xz^{\sigma})dz$$
$$=\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{x^k}{k!}\int_0^1 z^{a+\sigma k-1}(1-z)^{b-1}dz$$
$$=\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(a+\sigma k)\Gamma(b)}{\Gamma(a+b+\sigma k)}\frac{x^k}{k!}$$
$$=\frac{\Gamma(b)}{\Gamma(\delta)}\sum_{k=0}^{\infty}\frac{\Gamma[\delta+k,x]\Gamma(a+\sigma k)}{\Gamma(\alpha k+\beta)\Gamma(a+b+\sigma k)}\frac{x^k}{k!}$$
$$=\frac{\Gamma(b)}{\Gamma(\delta)}{}_2\overline{\Psi}_2\left[\begin{array}{cc} [\delta,1,x] & (a,\sigma) \\ (\beta,\alpha) & (a+b,\sigma)\end{array} \mid x\right]. \qquad (31)$$

Special Cases
(i) If $a=\beta$ and $\alpha=\sigma$ then (30) reduces to

$$\int_0^1 z^{\beta-1}(1-z)^{b-1}E_{\sigma,\beta}^{[\delta,x]}(xz^{\sigma})dz=\Gamma(b)E_{\sigma,\beta+b}^{[\delta,x]}(x) \qquad (32)$$

where $\delta,\beta,\sigma,b\in C$, $\Re(\beta)>0$, $\Re(\sigma)>0$, $\Re(b)>0$, $\Re(\delta)>0$.
(ii)

$$\int_0^1 z^{\sigma-1}(1-z)^{\beta-1}E_{\alpha,\beta}^{[\delta,x]}(x(1-z)^{\alpha})dz=\Gamma(\sigma)E_{\alpha,\sigma+\beta}^{[\delta,x]}(x), \qquad (33)$$

where $\alpha,\beta,\sigma,\delta\in C$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\sigma)>0$, $\Re(\delta)>0$.

(iii) If $\alpha=\beta=\delta=1$, then from (30),we obtain

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{1,1}^{[1,x]}(xz^{\sigma})dz=\Gamma(b)[e^{-x}e_n(x)]_1\Psi_1\left[\begin{array}{c}(a,\sigma)\\(a+b,\sigma)\end{array} \mid x\right], \qquad (34)$$

where $[1;x]_n=n!\,[e^{-x}e_n(x)]$ and $e_n(x)=\sum_{j=0}^{\infty}\frac{x^j}{j!}$, see [8].

7. Mellin Transforms of Incomplete Mittag-Leffler function

In this section we discuss about the Millen Barans integral representation of incomplete Mittag-Leffler function and with the help of its we will be find Millen transform of incomplete Mittag-Leffler function.

The Mellin transform (Sneddon [9]) of the function $f(z)$ is defined as

$$M[f(z);s] = \int_0^\infty z^{s-1} f(z)dz = f^*(s), \quad \Re(s) > 0 \tag{35}$$

and inverse mellin transform of the function $f(z)$ is defined as

$$f(z) = M^{-1}[f^*(s);x] = \frac{1}{2\pi i}\int_L f^*(s)x^{-s}ds. \tag{36}$$

Theorem 7.1. *If $\alpha, \beta, \delta, \in C$; $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0$ and $\delta \neq 0$ then incomplete Mittag-Leffler function $E_{\alpha,\beta}^{[\delta,x]}(z)$ and $E_{\alpha,\beta}^{(\delta,x)}(z)$ is represented in the Mellin-Barnes type integral as*

$$E_{\alpha,\beta}^{[\delta,x]} = \frac{1}{\Gamma(\delta)}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(\delta - s, x)}{\Gamma(\beta - \alpha s)}(-z)^{-s}ds, \tag{37}$$

and

$$E_{\alpha,\beta}^{(\delta,x)} = \frac{1}{\Gamma(\delta)}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\gamma(\delta - s, x)}{\Gamma(\beta - \alpha s)}(-z)^{-s}ds, \tag{38}$$

where $|arg(-z)| < \pi$ and the Barnes path of integration L start at $-i\infty$ and runs to $+i\infty$ in the s-plane, and the pole of $\Gamma(s)$ and $\Gamma(\delta - s, x)$ $\gamma(\delta - s, x)$ are sperate by the contour.

Proof. With the equation (6) and (7), we get

$$E_{\alpha,\beta}^{[\delta,x]}(z) = \frac{1}{\Gamma(\delta)}\sum_{k=o}^{\infty}\frac{\Gamma(\delta + k, x)}{\Gamma(\alpha k + \beta)}\frac{z^k}{k!}$$

$$= \frac{1}{\Gamma(\delta)}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(\delta - s, x)}{\Gamma(\beta - \alpha s)}(-z)^{-s}ds \tag{39}$$

and

$$E_{\alpha,\beta}^{(\delta,x)}(z) = \frac{1}{\Gamma(\delta)}\sum_{k=o}^{\infty}\frac{\gamma(\delta + k, x)}{\Gamma(\alpha k + \beta)}\frac{z^k}{k!}$$

$$= \frac{1}{\Gamma(\delta)}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\gamma(\delta - s, x)}{\Gamma(\beta - \alpha s)}(-z)^{-s}ds \tag{40}$$

which is required result.

Theorem 7.2. *If $\alpha, \beta, \delta, s \in C$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$ and $\Re(\delta) > 0$, then*

$$\int_0^\infty t^{s-1}E_{\alpha,\beta}^{[\delta,x]}(-zt)dt = \frac{\Gamma(s)\Gamma(\delta - s, x)}{z^s\Gamma(s)\Gamma(\beta - \alpha s)}. \tag{41}$$

Proof. With equation (37) $E^{[\delta,x]}_{\alpha,\beta}(-zt)$ and can be written as

$$E^{[\delta,x]}_{\alpha,\beta}(-zt) = \frac{1}{2\pi i\Gamma(\delta)}\int_L \frac{\Gamma(s)\Gamma(\delta-s,x)}{\beta-\alpha s}(zt)^{-s}ds$$

$$= \frac{1}{2\pi i\Gamma(\delta)}\int_L f^*(s)t^{-s}ds, \tag{42}$$

where

$$f^*(s) = \frac{\Gamma(s)\Gamma(\delta-s,x)}{z^s\Gamma(\delta)\Gamma(\beta-\alpha s)}.$$

Then with the help of (35), (36) and (42) we arrive at desired result.

8. Basic Properties of the Incomplete Mittag-Leffler Function

Theorem 8.1. *If* $\alpha,\beta,\delta,\in C$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ *for* $m\in N$

$$\left(\frac{d}{dz}\right)^m E^{[\delta,x]}_{\alpha,\beta}(z) = (\delta)_m E^{[\delta+m,x]}_{\alpha,\alpha m+\beta}(z) \tag{43}$$

and

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}E^{[\delta,x]}_{\alpha,\beta}(\omega z^\alpha)\right] = z^{\beta-m-1}E^{[\delta,x]}_{\alpha,\beta-m}(\omega z^\alpha). \tag{44}$$

Proof.

$$\left(\frac{d}{dz}\right)^m E^{[\delta,x]}_{\alpha,\beta}(z) = \left(\frac{d}{dz}\right)^m \sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}$$

$$= \sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{1}{k!}\left(\frac{d}{dz}\right)^m z^k$$

$$= \sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{1}{k!}\frac{\Gamma(k+1)}{\Gamma(k-m+1)}z^{k-m}$$

$$= \sum_{k=m}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{1}{(k-m)!}z^{k-m}$$

$$= \sum_{k=0}^{\infty}\frac{[\delta,x]_{k+m}}{\Gamma(\alpha(k+m)+\beta)}\frac{1}{k!}z^k$$

$$= (\delta)_m E^{[\delta+m,x]}_{\alpha,\alpha m+\beta}(z) \tag{45}$$

and for (44)

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{[\delta,x]}_{\alpha,\beta}(\omega z^\alpha)\right]$$
$$= \sum_{k=0}^{\infty} \frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)} \frac{\omega^k}{k!} \left(\frac{d}{dz}\right)^m z^{\alpha k+\beta-1}$$
$$= z^{\beta-m-1} \sum_{k=0}^{\infty} \frac{[\delta,x]_k}{\Gamma(\alpha k+\beta-m)} \frac{(\omega z^\alpha)^k}{k!} \tag{46}$$

which is required result.

Theorem 8.2. *If $\alpha, \beta, \delta \in C$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$ then*

$$\frac{1}{\Gamma(\delta)} \int_0^1 x^{\beta-1}(1-x)^{\delta-1} E^{[\delta,x]}_{\alpha,\beta}(x^\alpha z)dx = E^{[\delta,x]}_{\alpha,\beta+\delta}(z) \tag{47}$$

Proof.

$$\frac{1}{\Gamma(\delta)} \int_0^1 x^{\beta-1}(1-x)^{\delta-1} E^{[\delta,x]}_{\alpha,\beta}(x^\alpha z)dx$$
$$= \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} \int_0^1 x^{\beta-1}(1-x)^{\delta-1} x^{\alpha k} dx$$
$$= \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} \frac{\Gamma(\alpha k+\beta)\Gamma(\delta)}{\Gamma(\alpha k+\beta+\delta)}$$
$$= \sum_{k=0}^{\infty} \frac{[\delta,x]_k}{\Gamma(\alpha k+\beta+\delta)} \frac{z^k}{k!}$$
$$= E^{[\delta,x]}_{\alpha,\beta+\delta}(z). \tag{48}$$

Theorem 8.3. *If $\alpha, \beta, \delta \in C$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$ then*

$$\frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{E^{[\delta,x]}_{\alpha,\beta}(\lambda t^\alpha)dt}{(x-t)^{1-\alpha}} = \frac{(\lambda x^\alpha)^k}{\Gamma(\delta)} {}_2\overline{\Psi}_2 \left[\begin{matrix} [\delta,1,x], (1,\alpha) \\ (\beta,\alpha)\,(\alpha+1,\alpha) \end{matrix} \mid \lambda x^\alpha \right] \tag{49}$$

Proof.

$$\begin{aligned}
&\frac{\lambda}{\Gamma(\alpha)}\int_0^x \frac{E_{\alpha,\beta}^{[\delta,x]}(\lambda t^\alpha)dt}{(x-t)^{1-\alpha}} \\
&= \frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{(\lambda t^\alpha)^k}{k!}dt \\
&= \frac{\lambda}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{\lambda^k}{k!}\int_0^x (x-t)^{\alpha-1}t^{\alpha k}dt \\
&= \frac{\lambda}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{\lambda^k}{k!}\int_0^x x^{\alpha-1}\left(1-t/x\right)^{\alpha-1}t^{\alpha k}dt \\
&= \frac{\lambda}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta)}\frac{\lambda^k}{k!}x^{\alpha(1+k)}\frac{\Gamma(\alpha k+1)\Gamma(\alpha)}{\Gamma(\alpha+\alpha k+1)} \\
&= \frac{\lambda x^\alpha}{\Gamma(\delta)}\sum_{k=0}^{\infty}\frac{[\delta+k,x]\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta)\Gamma(\alpha k+\alpha+1)}\frac{(\lambda x^\alpha)^k}{k!}
\end{aligned}$$

$$= \frac{(\lambda_x^\alpha)^k}{\Gamma(\delta)}{}_2\overline{\Psi}_2\left[\begin{array}{c}[\delta,1,x]\,,\,(1,\alpha)\\(\beta,\alpha)\,(\alpha+1,\alpha)\end{array}\mid \lambda x^\alpha.\right] \tag{50}$$

Theorem 8.4. *If* $\alpha,\beta,\delta\in C$, $\Re(\alpha)>0$, $\Re(\beta)>0$ $\Re(\delta)>0$ *then*

$$E_{\alpha,\beta}^{[\delta,x]}(z) = \beta E_{\alpha,\beta+1}^{[\delta,x]}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{[\delta,x]}(z) \tag{51}$$

Proof.

$$\begin{aligned}
&\beta E_{\alpha,\beta+1}^{[\delta,x]}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{[\delta,x]}(z) \\
&= \beta E_{\alpha,\beta+1}^{[\delta,x]}(z) + \alpha z\frac{d}{dz}\sum_{k=0}^{\infty}\frac{[\delta,x]_k}{\Gamma(\alpha k+\beta+1)}\frac{z^k}{k!} \\
&= \beta E_{\alpha,\beta+1}^{[\delta,x]}(z) + \sum_{k=0}^{\infty}\frac{\alpha k[\delta,x]_k}{\Gamma(\alpha k+\beta+1)}\frac{z^k}{k!}
\end{aligned}$$

$$= E_{\alpha,\beta}^{[\delta,x]}(z). \tag{52}$$

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Dhamendra Kumar Singh
Department of Mathematics
University of Institute Engineering and Technology
CSJM University, Kanpur-208024, (U.P.), India
email:*drdksinghabp@gmail.com*

Saurabh Porwal
Department of Mathematics
University of Institute Engineering and Technology
CSJM University, Kanpur-208024, (U.P.), India
email:*saurabhjcb@rediffmail.com*