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ON COMPLETE SPACELIKE HYPERSURFACES IN ANTI-DE SITTER SPACE $H_1^{N+1}(-1)$

Yingbo Han and Shuxiang Feng

Abstract. In this paper, we investigate complete spacelike hypersurfaces with constant mean curvature in anti-de Sitter space $H_1^{n+1}(-1)$. Some rigidity theorems are obtained for these hypersurfaces.

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1. Introduction

Let $M_1^{n+1}(c)$ denote an $(n+1)$-dimensional Lorentzian manifold of constant curvature c, which is called a Lorentzian space form.Then an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ is said to be a de Sitter space $S_1^{n+1}(c)$, a Lorentzian Minkowski space L^{n+1} or an anti-de Sitter space $H_1^{n+1}(c)$ respectively, according to its sectional curvature $c > 0, c = 0$ or $c < 0$. A hypersuface M in a Lorentizian space form $M_1^{n+1}(c)$ is said to be spacelike if the induced metric on M from that of $M_1^{n+1}(c)$ is positive definite.

In recent years, the study of spacelike hypersurfaces in semi-Riemannian ambients has got increasing interesting motivated by their importance in problems related to Physics, more specifically in the theory of general relativity.

E.Calabi [1] first studied the Bernstein problem for maximal spacelike entire graphs in R_1^{n+1}, $n \leq 4$, and proved that it must be hyperplane. Later S.Y. Cheng and S.T. Yau [2] showed that this conclusion remains true for arbitrary n. In [4] T. Ishihara proved that complete maximal spacelike hypersurfaces of $M_1^{n+1}(c)$, $c \geq 0$, are totally geodesic. Further, in the same paper, T. Ishihara also proved the following result:

Theorem 1.1.[4]. Let M^n be an n-dimensional complete maximal spacelike hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$, then the norm square of the second fundamental form of M satisfies $S \leq n$ and $S = n$ if and only if $M^n = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$.

In [3], L.F. Cao and G.X. Wei gave a new characterization of hyperbolic cylinder $M^n = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$ in anti-de Sitter space $H_1^{n+1}(-1)$.

Theorem 1.2.[3]. Let M^n be an n-dimensional ($n \geq 3$) complete maximal spacelike hyperusrface with two distinct principal curvature λ and μ in anti-de Sitter space$H_1^{n+1}(-1)$. If $\inf|\lambda-\mu|>0$, then $M^n = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$.

In [5], C.X.Nie studied complete spacelike hypersurfaces with constant mean curvature in anti-de Sitter space $H_1^{n+1}(-1)$ and gave the following result:

Theorem 1.3.[5]. Let M^n be an n-dimensional ($n \geq 3$) complete spacelike hyperusrface with constant mean curvature and two distinct principal curvature λ and μ in anti-de Sitter space$H_1^{n+1}(-1)$. If $\inf|\lambda-\mu|>0$, then $M^n = H^m(-\frac{1}{a^2}) \times H^{n-m}(-\frac{1}{1-a^2})$, $(1 \leq m \leq n-1)$.

In this note, we also investigate complete spacelike hypersurfaces with constant mean curvature in $H_1^{n+1}(-1)$. More precisely, we prove the following results:

Theorem 1.4. Let M^n $(n \geq 3)$ be a complete spacelike hypersurface with constant mean curvature H in $H_1^{n+1}(-1)$. Assume that M^n has $n-1$ principal curvatures with the same sign everywhere. If the Ricci curvature Ric_M of M^n and S satisfy the following:

$$\begin{aligned} Ric_M &\geq -\frac{n(n-2)}{n-1}[1+\frac{n^2H^2}{2(n-1)}-\frac{\sqrt{n^2H^4+4(n-1)H^2}}{2(n-1)}] = -C_-(H) \\ S &\leq n+\frac{n^3H^2}{2(n-1)}+\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2} = S_+(H), \end{aligned}$$

then S is constant, $S = S_+(H)$ and $M^n = H^1(-\frac{1}{a^2}) \times H^{n-1}(-\frac{1}{1-a^2})$ with $a^2 \leq \frac{1}{n}$.

Corollary 1.5. Let M^n $(n \geq 3)$ be a complete maximal spacelike hypersurface in $H_1^{n+1}(-1)$. Assume that M^n has $n-1$ principal curvatures with the same sign everywhere. If $Ric_M \geq -\frac{n(n-2)}{n-1}$, then $S = n$ and $M^n = H^1(-n) \times H^{n-1}(-\frac{n}{n-1})$.

Theorem 1.6. Let M^n $(n \geq 3)$ be a complete spacelike hypersurface with constant mean curvature H in $H_1^{n+1}(-1)$. Assume that M^n has $n-1$ principal curvatures with the same sign everywhere. If $-C_-(H) \leq Ric_M \leq 0$, then S is constant, $S = S_+(H)$ and $M^n = H^1(-\frac{1}{a^2}) \times H^{n-1}(-\frac{1}{1-a^2})$ with $a^2 \leq \frac{1}{n}$.

2.Preliminaries

Let M^n be an n-dimensional spacelike hypersurface of $H_1^{n+1}(-1)$. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \cdots, e_n, e_{n+1}\}$ in $H_1^{n+1}(-1)$ such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+1}$ be its dual frame field such that the semi-Riemannian metric of $H_1^{n+1}(c)$ is given by $ds^2 = \sum_{A=1}^{n+1} \epsilon_A(\omega_A)^2$, where $\epsilon_i = 1$, $i = 1, \cdots, n$ and $\epsilon_{n+1} = -1$. Then the structure equations of $S_1^{n+1}(1)$ are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D, \tag{2}$$

$$K_{ABCD} = -\epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{3}$$

We restrict these forms to M^n, then $\omega_{n+1} = 0$ and the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since

$$0 = d\omega_{n+1} = \sum_i \omega_{n+1,i} \wedge \omega_i, \tag{4}$$

by Cartan's lemma we may write

$$\omega_{n+1,i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{5}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$
$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{6}$$

where R_{ijkl} are the components of curvature tensor of M^n. We call

$$B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \otimes e_{n+1} \tag{7}$$

the second fundamental form of M^n.

From the above equation, we have

$$R = -n(n-1) - n^2H^2 + S, \tag{8}$$

where R is the scalar curvature and S is the norm square of the second fundamental form and H is the mean curvature, then we have

$$S = \sum_{ij} h^2, \quad H = \frac{1}{n} \sum_i h_{ii}.$$

Now, we compute some local formulas. For any fixed point x in M, we can choose a local frame field $\{e_1, \cdots, e_n\}$, such that

$$h_{ij}(x) = \lambda_i(x)\delta_{ij}, \quad i, j = 1, \cdots, n.$$

where λ_i are principal curvatures.

Example 1. Let $M = H^1(-\frac{1}{a^2}) \times H^{n-1}(-\frac{1}{1-a^2})$ $(a > 0)$ be a spacelike hypersurface of $H_1^{n+1}(-1)$. Then M has two distinct constant principal curvatures

$$\lambda_1 = \frac{\sqrt{1-a^2}}{a}, \quad \lambda_2 = \cdots = \lambda_n = -\frac{a}{\sqrt{1-a^2}}.$$

and constant mean curvature $H = \frac{1}{n}\sum \lambda_i = \frac{1-na^2}{na\sqrt{1-a^2}}$.

If $a^2 < \frac{1}{n}$, then we have

$$S = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2} = S_+(H)$$

and the infremum of Ricci curvature of M^n is given by

$$-C_-(H) = -\frac{n(n-2)}{n-1}[1 + \frac{n^2H^2}{2(n-1)} - \frac{\sqrt{n^2H^4 + 4(n-1)H^2}}{2(n-1)}]$$

If $a^2 = \frac{1}{n}$, then we have $H = 0$, $S = n$
and the infremum of Ricci curvature of M^n is given by $-\frac{n(n-2)}{n-1}$.
If $1 > a^2 > \frac{1}{n}$, then we have

$$S = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2} = S_-(H)$$

and the infremum of Ricci curvature of M^n is given by

$$-C_+(H) = -\frac{n(n-2)}{n-1}[1 + \frac{n^2H^2}{2(n-1)} + \frac{\sqrt{n^2H^4 + 4(n-1)H^2}}{2(n-1)}].$$

3.Proof of Theorems

By renumbering the principal directions $e_1, \cdots, e_n$, if necessary, we may assume that the principal curvature satisfy

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$$

Then we have

$$\begin{aligned} S &= \sum_{i=1}^{n}\lambda_i^2, \quad nH=\sum_i \lambda_i && (9)\\ R_{ijkl} &= -\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}-\lambda_i\lambda_j\delta_{ik}\delta_{jl}+\lambda_i\lambda_j\delta_{il}\delta_{jk} && (10)\\ Ric_{ii} &= \sum_{k=1}^{n}R_{ikik}=-(n-1)-nH\lambda_i+\lambda_i^2 && (11)\end{aligned}$$

Set

$$P(t)=t^2-nHt-(n-1), \tag{12}$$

It has two real roots $\Lambda_\pm=\frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$. From (11) and (12), we have

$$Ric_{ii}=P(\lambda_i). \tag{13}$$

In the next part, we give the proof of Theorem 1.4.

Proof of Theorem 1.4:

Assume $H\geq 0$. From (1) and (8), we have

$$\begin{aligned} R &= -n(n-1)-n^2H^2+S\\ &\leq -n(n-1)-n^2H^2+n+\frac{n^3H^2}{2(n-1)}+\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}\\ &= -n(n-2)[1+\frac{n^2H^2}{2(n-1)}-\frac{\sqrt{n^2H^4+4(n-1)H^2}}{2(n-1)}]\\ &= -(n-1)C_-(H).\end{aligned}$$

By using the conditions $R=\sum_i Ric_{ii}$ and $Ric_{ii}\geq -C_-(H)$, we have $Ric_{ii}\leq 0$ for $i\in\{1,\cdots,n\}$. From (13), we have

$$P(\lambda_i)\leq 0,$$

for $i=1,\cdots,n$. So we have

$$\Lambda_-\leq\lambda_n\leq\lambda_{n-1}\leq\cdots\leq\lambda_1\leq\Lambda_+.$$

Denote $\mu=\frac{nH-\sqrt{n^2H^2+4(n-1)}}{2(n-1)}$, we have $P(\mu)=P(nH-\mu)=-C_-(H)$. Since M^n has $(n-1)$ principal curvatures with the same sign everywhere and $Ric_{ii}\geq -C_-(H)$, then we have the following possible case.

Case A:

$$\Lambda_-\leq\lambda_n\leq\lambda_{n-1}\leq\cdots\leq\lambda_2\leq\mu<0<nH-\mu\leq\lambda_1\leq\Lambda_+.$$

Case B:

$$\Lambda_- \leq \lambda_n \leq \mu < 0 < nH - \mu \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq \Lambda_+.$$

If the principal curvatures satisfy Case A, then we have

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \mu < 0,$$

On the other hand, we have

$$\sum_{i=2}^{n} \lambda_i = nH - \lambda_1 \geq nH - \Lambda_+ = \frac{nH - \sqrt{n^2H^2 + 4(n-1)}}{2} = (n-1)\mu.$$

So we have

$$\lambda_n = \cdots = \lambda_2 = \mu, \quad \lambda_1 = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2},$$

$$S = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$$

and

$$\inf |\lambda_1 - \lambda_2| = \frac{(2n-3)nH + n\sqrt{n^2H^2 + 4(n-1)}}{2(n-1)} > 0.$$

then from Theorem 1.3 and Example 1, we know that $M^n = H^1(-\frac{1}{a^2}) \times H^{n-1}(-\frac{1}{1-a^2})$ with $a^2 \leq \frac{1}{n}$.

If the principal curvatures satisfy Case B, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} \lambda_i &= nH - \lambda_n \\ &\leq nH - \frac{nH - \sqrt{n^2H^2 + 4(n-1)}}{2} \\ &= \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2}. \end{aligned} \tag{14}$$

On other hand, we have

$$\sum_{i=1}^{n-1} \lambda_i \geq (n-1)(nH - \mu) = \frac{(2n-3)nH + \sqrt{n^2H^2 + 4(n-1)}}{2} \tag{15}$$

From (14) and (15), we have

$$\frac{(2n-3)nH+\sqrt{n^2H^2+4(n-1)}}{2} \leq \frac{nH+\sqrt{n^2H^2+4(n-1)}}{2},$$

so

$$H \leq 0.$$

Since $H \geq 0$, then $H = 0$. So the case B turns into the following:

$$-\sqrt{n-1} \leq \lambda_n \leq -\frac{1}{\sqrt{n-1}} < 0 < \frac{1}{\sqrt{n-1}} \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \leq \sqrt{n-1}. \quad (16)$$

then we have

$$(n-1)\frac{1}{\sqrt{n-1}} = \sqrt{n-1} \leq \sum_{i=1}^{n-1} \lambda_i = -\lambda_n \leq \sqrt{n-1}. \quad (17)$$

From (16) and (17), we have

$$\lambda_1 = \cdots = \lambda_{n-1} = \frac{1}{\sqrt{n-1}}$$

and

$$\lambda_n = -\sqrt{n-1}.$$

So

$$S = \sum_{i=1}^{n} \lambda_i^2 = n$$

From Theorem 1.1 and $S = n$, we know that $M^n = H^1(-n) \times H^{n-1}(-\frac{n}{n-1})$. Thus we complete the proof of Theorem 1.4.

Proof of Corollary 1.5: Since M^n is a complete maximal spacelike hypersurface of $H_1^{n+1}(-1)$, then we know that $S \leq n$ from Theorem 1.1. So we know that M^n satisfies the following:

$$Ric_M \geq -\frac{n(n-2)}{n-1}$$

and

$$S \leq n.$$

From Theorem 1.4, we know that S is constant, $S = n$ and $M^n = H^1(-n) \times H^{n-1}(-\frac{n}{n-1})$. This completes the proof of Corollary 1.5.

Proof of Theorem 1.6: Since $-C_-(H) \leq Ric_M \leq 0$, then we have

$$-C_-(H) \leq P(\lambda_i) = \lambda_i^2 - nH\lambda_i - (n-1) \leq 0$$

So we know that the principal curvatures satisfy the Case A or Case B. From the proof of Theorem 1.4, we know that Theorem 1.6 is true.

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Yingbo Han and Shuxiang Feng
College of Mathematics and Information Science
Xinyang Normal University
Xinyang, 464000, Henan, P. R. China
email:*hyb@xynu.edu.cn; shuxiangfeng78@163.com*