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A NOTE ON STRONG DIFFERENTIAL SUBORDINATIONS USING A GENERALIZED SĂLĂGEAN OPERATOR AND RUSCHEWEYH OPERATOR

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Abstract. In the present paper we establish several strong differential subordinations regardind the extended new operator DR_λ^m defined by the Hadamard product of the extended generalized Sălăgean operator D_λ^m and the extended Ruscheweyh derivative R^m, given by $DR_\lambda^m : \mathcal{A}_{n\zeta}^* \to \mathcal{A}_{n\zeta}^*$, $DR_\lambda^m f(z,\zeta) = (D_\lambda^m * R^m) f(z,\zeta)$, where $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$ is the class of normalized analytic functions.

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1. Introduction

Denote by U the unit disc of the complex plane $U = \{z \in C : |z| < 1\}$, $\overline{U} = \{z \in C : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\},$$

where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\},$$

for $a \in C$ and $n \in N$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [7] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [9], [8].

Definition No. 1 *[9] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark No. 1 [9] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1 is equivalent to $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

We have need the following lemmas to study the strong differential subordinations.

Lemma No. 1 *[4] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in C^*$ be a complex number with $Re\gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta]$ and*

$$p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta) \prec\prec h(z,\zeta),$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

where $g(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{\gamma}{n}-1} dt$ is convex and it is the best dominant.

Lemma No. 2 *[4] Let $g(z,\zeta)$ be a convex function in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and let*

$$h(z,\zeta) = g(z,\zeta) + n\alpha z g'_z(z,\zeta), \quad z \in U, \zeta \in \overline{U},$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z,\zeta) = g(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots, \quad z \in U, \zeta \in \overline{U},$$

is holomorphic in $U \times \overline{U}$ and

$$p(z,\zeta) + \alpha z p'_z(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U},$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta)$$

and this result is sharp.

We also extend the generalized Sălăgean differential operator [6] and Ruscheweyh derivative [10] to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$ introduced in [8].

Definition No. 2 *[5] For $f \in \mathcal{A}^*_{n\zeta}$, $\lambda \geq 0$ and $n, m \in N$, the extended operator D^m_λ is defined by $D^m_\lambda : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned}
D^0_\lambda f(z,\zeta) &= f(z,\zeta) \\
D^1_\lambda f(z,\zeta) &= (1-\lambda) f(z,\zeta) + \lambda z f'(z,\zeta) = D_\lambda f(z,\zeta) \\
&\ldots \\
D^{m+1}_\lambda f(z,\zeta) &= (1-\lambda) D^m_\lambda f(z,\zeta) + \lambda z \left(D^m_\lambda f(z,\zeta)\right)' = D_\lambda \left(D^m_\lambda f(z,\zeta)\right), z \in U, \zeta \in \overline{U}.
\end{aligned}$$

Remark No. 2 If $f \in \mathcal{A}^*_{n\zeta}$ and $f(z) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $D^m_\lambda f(z,\zeta) = z + \sum_{j=n+1}^{\infty} [1+(j-1)\lambda]^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Definition No. 3 *[5] For $f \in \mathcal{A}^*_{n\zeta}$, $n, m \in N$, the extended operator R^m is defined by $R^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta) \\ R^1 f(z,\zeta) &= z f'(z,\zeta) \\ &\ldots \\ (m+1) R^{m+1} f(z,\zeta) &= z\left(R^m f(z,\zeta)\right)' + m R^m f(z,\zeta), z \in U, \zeta \in \overline{U}. \end{aligned}$$

Remark No. 3 If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $R^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

We extend the differential operator studied in [1], [2] to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$.

Definition No. 4 *Let $\lambda \geq 0$ and $m \in N \cup \{0\}$. Denote by DR^m_λ the extended operator given by the Hadamard product (the convolution product) of the extended generalized Sălăgean operator D^m_λ and the extended Ruscheweyh operator R^m, $DR^m_\lambda : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$DR^m_\lambda f(z,\zeta) = \left(D^m_\lambda * R^m\right) f(z,\zeta).$$

Remark No. 4 If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $DR^m_\lambda f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} [1+(j-1)\lambda]^m a_j^2(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Remark No. 5 For $\lambda = 1$ we obtain the Hadamard product SR^m [3] of the extended Sălăgean operator S^m and the extended Ruscheweyh derivative R^m.

2.Main results

Definition No. 5 *Let $\delta \in [0,1)$, $\lambda \geq 0$ and $m \in N$. A function $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ is said to be in the class $\mathcal{DR}_m(\delta,\lambda,\zeta)$ if it satisfies the inequality*

$$\text{Re}\ \left(DR^m_\lambda f(z,\zeta)\right)'_z > \delta, \quad z \in U, \zeta \in \overline{U}. \tag{1}$$

Theorem No. 1 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+\frac{1}{c+2}zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, $c>0$. If $\lambda\geq 0$, $n,m\in N$, $f\in\mathcal{DR}_m(\delta,\lambda,\zeta)$ and $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, then*

$$\left(DR_\lambda^m f(z,\zeta)\right)'_z \prec\prec h(z,\zeta)\,, z\in U,\zeta\in\overline{U}, \tag{2}$$

implies

$$\left(DR_\lambda^m F(z,\zeta)\right)'_z \prec\prec g(z,\zeta)\,, z\in U,\zeta\in\overline{U},$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F(z,\zeta)=(c+2)\int_0^z t^c f(t,\zeta)\,dt. \tag{3}$$

Differentiating (3), with respect to z, we have $(c+1)F(z,\zeta)+zF'_z(z,\zeta)=(c+2)f(z,\zeta)$ and

$$(c+1)\,DR_\lambda^m F(z,\zeta)+z\left(DR_\lambda^m F(z,\zeta)\right)'_z=(c+2)\,DR_\lambda^m f(z,\zeta)\,,\quad z\in U,\zeta\in\overline{U}. \tag{4}$$

Differentiating (4) with respect to z we have

$$\left(DR_\lambda^m F(z,\zeta)\right)'_z+\frac{1}{c+2}z\left(DR_\lambda^m F(z,\zeta)\right)''_{z^2}=\left(DR_\lambda^m f(z,\zeta)\right)'_z,\quad z\in U,\zeta\in\overline{U}. \tag{5}$$

Using (5), the strong differential subordination (2) becomes

$$\left(DR_\lambda^m F(z,\zeta)\right)'_z+\frac{1}{c+2}z\left(DR_\lambda^m F(z,\zeta)\right)''_{z^2}\prec\prec g(z,\zeta)+\frac{1}{c+2}zg'_z(z,\zeta)\,. \tag{6}$$

Denote

$$p(z,\zeta)=\left(DR_\lambda^m F(z,\zeta)\right)'_z,\quad z\in U,\zeta\in\overline{U}. \tag{7}$$

Replacing (7) in (6) we obtain

$$p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)\prec\prec g(z,\zeta)+\frac{1}{c+2}zg'_z(z,\zeta)\,, z\in U,\zeta\in\overline{U}.$$

Using Lemma 2 we have

$$p(z,\zeta)\prec\prec g(z,\zeta)\,, z\in U,\zeta\in\overline{U}, i.e.\left(DR_\lambda^m F(z,\zeta)\right)'_z\prec\prec g(z,\zeta)\,, z\in U,\zeta\in\overline{U},$$

and this result is sharp.

Thoerem No. 2 *Let* $h(z,\zeta)=\frac{\zeta+(2\delta-\zeta)z}{1+z}$, $z\in U$, $\zeta\in\overline{U}$, $\delta\in[0,1)$ *and* $c>0$. *If* $\lambda\geq 0$, $m\in N$ *and* I_c *is given by Theorem 1, then*

$$I_c\left[\mathcal{DR}_m(\delta,\lambda,\zeta)\right]\subset\mathcal{DR}_m(\delta^*,\lambda,\zeta), \tag{8}$$

where $\delta^*=2\delta-\zeta+\frac{2(c+2)(\zeta-\delta)}{n}\beta\left(\frac{c+2}{n}-2\right)$ *and* $\beta(x)=\int_0^1\frac{t^{x+1}}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 1 we get from the hypothesis of Theorem 2 that

$$p(z,\zeta)+\frac{1}{c+2}zp_z'(z,\zeta)\prec h(z,\zeta),$$

where $p(z,\zeta)$ is defined in (7).

Using Lemma 1 for $\gamma=c+2$, we deduce that

$$p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),$$

that is

$$\left(DR_\lambda^m F(z,\zeta)\right)_z'\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),$$

where

$$\begin{aligned} g(z,\zeta) &= \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z t^{\frac{c+2}{n}-1}\frac{\zeta+(2\delta-\zeta)t}{1+t}dt= \\ & (2\delta-\zeta)+\frac{2(c+2)(\zeta-\delta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{1+t}dt. \end{aligned}$$

Since g is convex and $g\left(U\times\overline{U}\right)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned} \text{Re }\left(DR_\lambda^m F(z,\zeta)\right)_z' \geq \min_{|z|=1}\text{Re }g(z,\zeta)=\text{Re }g(1,\zeta)=\delta^*= \\ 2\delta-\zeta+\frac{2(c+2)(\zeta-\delta)}{n}\beta\left(\frac{c+2}{n}-2\right). \end{aligned} \tag{9}$$

From (9) we deduce inclusion (8).

Theorem No. 3 *Let* $g(z,\zeta)$ *be a convex function,* $g(0,\zeta)=1$ *and let* h *be the function* $h(z,\zeta)=g(z,\zeta)+zg_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. *If* $\lambda\geq 0$, $m\in N\cup\{0\}$, $f\in\mathcal{A}_{n\zeta}^*$ *and verifies the strong differential subordination*

$$\left(DR_\lambda^m f(z,\zeta)\right)_z'\prec\prec h(z,\zeta),\quad z\in U,\zeta\in\overline{U}, \tag{10}$$

then

$$\frac{DR_{\lambda}^{m}f(z,\zeta)}{z} \prec\prec g(z,\zeta), z \in U, \zeta \in \overline{U},$$

and this result is sharp.

Proof. For $f \in \mathcal{A}_{n\zeta}^{*}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$ we have $DR_{\lambda}^{m}f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^m a_j^2(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = \frac{DR_{\lambda}^{m}f(z,\zeta)}{z} = \frac{z+\sum_{j=n+1}^{\infty} C_{m+j-1}^{m}[1+(j-1)\lambda]^m a_j^2(\zeta) z^j}{z} = 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^m a_j^2(\zeta) z^{j-1}$.

We have $p(z,\zeta) + zp'_z(z,\zeta) = (DR_{\lambda}^{m}f(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$.

Then $(DR_{\lambda}^{m}f(z,\zeta))'_z \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, becomes $p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 2 we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{DR_{\lambda}^{m}f(z,\zeta)}{z} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Theorem No. 4 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. If $\lambda \geq 0$, $m \in N \cup \{0\}$, $f \in \mathcal{A}_{n\zeta}^{*}$ and verifies the strong differential subordination*

$$(DR_{\lambda}^{m}f(z,\zeta))'_z \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}, \tag{11}$$

then

$$\frac{DR_{\lambda}^{m}f(z,\zeta)}{z} \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), z \in U, \zeta \in \overline{U},$$

where $g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$ is convex and it is the best dominant.

Proof. For $f \in \mathcal{A}_{n\zeta}^{*}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$ we have $DR_{\lambda}^{m}f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^m a_j^2(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = \frac{DR_{\lambda}^{m}f(z,\zeta)}{z} = \frac{z+\sum_{j=n+1}^{\infty} C_{m+j-1}^{m}[1+(j-1)\lambda]^m a_j^2(\zeta) z^j}{z} = 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^m a_j^2(\zeta) z^{j-1} \in \mathcal{H}^*[1,n,\zeta]$.

We have $p(z,\zeta) + zp'_z(z,\zeta) = (DR_{\lambda}^{m}f(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$.

Then $(DR_{\lambda}^{m}f(z,\zeta))'_z \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, becomes $p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1 for $\gamma = 1$, we obtain $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{DR_{\lambda}^{m}f(z,\zeta)}{z} \prec\prec g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant.

Corollary No. 1 *Let $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ a convex function in $U \times \overline{U}$, $0 \leq \beta < 1$. If $\lambda \geq 0$, $m,n \in N$, $f \in \mathcal{A}_{n\zeta}^{*}$ and verifies the strong differential subordination*

$$(DR_{\lambda}^{m}f(z,\zeta))'_z \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}, \tag{12}$$

then

$$\frac{DR_{\lambda}^{m}f(z,\zeta)}{z} \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), z \in U, \zeta \in \overline{U},$$

where g *is given by* $g(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z \in U$, $\zeta \in \overline{U}$. *The function* g *is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 4 and considering $p(z,\zeta) = \frac{DR_{\lambda}^{m}f(z,\zeta)}{z}$, the strong differential subordination (12) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}, \quad z \in U, \zeta \in \overline{U}.$$

By using Lemma 1 for $\gamma = 1$, we have $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e.

$$\frac{DR_{\lambda}^{m}f(z,\zeta)}{z} \prec\prec g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1}dt = \frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{\zeta + (2\beta-\zeta)\,t}{1+t}dt$$

$$= 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt, \quad z \in U, \zeta \in \overline{U}.$$

Theorem No. 5 *Let* $g(z,\zeta)$ *be a convex function such that* $g(0,\zeta) = 1$ *and let* h *be the function* $h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. *If* $\lambda \geq 0$, $m \in N \cup \{0\}$, $f \in \mathcal{A}^*_{n\zeta}$ *and verifies the strong differential subordination*

$$\left(\frac{zDR_{\lambda}^{m+1}f(z,\zeta)}{DR_{\lambda}^{m}f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U}, \tag{13}$$

then

$$\frac{DR_{\lambda}^{m+1}f(z,\zeta)}{DR_{\lambda}^{m}f(z,\zeta)} \prec\prec g(z,\zeta), z \in U, \zeta \in \overline{U},$$

and this result is sharp.

Proof. For $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta)\, z^j$ we have $DR_{\lambda}^{m}f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m}\left[1 + (j-1)\lambda\right]^m a_j^2(\zeta)\, z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = \frac{DR_{\lambda}^{m+1}f(z,\zeta)}{DR_{\lambda}^{m}f(z,\zeta)} = \frac{z+\sum_{j=n+1}^{\infty} C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2(\zeta)z^j}{z+\sum_{j=n+1}^{\infty} C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2(\zeta)z^j} = \frac{1+\sum_{j=n+1}^{\infty} C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2(\zeta)z^{j-1}}{1+\sum_{j=n+1}^{\infty} C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2(\zeta)z^{j-1}}$.

We have $p'_z(z,\zeta) = \frac{\left(DR_\lambda^{m+1} f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)} - p(z,\zeta) \cdot \frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}$.

Then $p(z,\zeta) + zp'_z(z,\zeta) = \left(\frac{zDR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)'_z$.

Relation (13) becomes $p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and by using Lemma 2 we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{DR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Theorem No. 6 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + \frac{n\lambda}{m\lambda+1} zg'_z(z,\zeta)$, $z \in U$, $\zeta \in U$, $\lambda \geq 0$, $m,n \in N$. If $f \in \mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U},$$

holds, then

$$\left(DR_\lambda^m f(z,\zeta)\right)'_z \prec\prec g(z,\zeta), z \in U, \zeta \in \overline{U},$$

and this result is sharp.

Proof. With notation
$p(z,\zeta) = \left(DR_\lambda^m f(z,\zeta)\right)'_z = 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m a_j^2(\zeta) z^{j-1}$ and $p(0,\zeta) = 1$, we obtain for $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$,

$p(z,\zeta) + zp'_z(z,\zeta) = 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m j^2 a_j^2(\zeta) z^{j-1} =$
$\frac{m+1}{\lambda z}\left[z + \sum_{j=n+1}^{\infty} C_{m+j}^{m+1} [1+(j-1)\lambda]^{m+1} a_j^2(\zeta) z^j\right] + \frac{\lambda-m-1}{\lambda} -$
$\sum_{j=n+1}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m a_j^2(\zeta) z^{j-1}\left(m-1+\frac{1}{\lambda}\right) j -$
$\sum_{j=n+1}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m a_j^2(\zeta) z^{j-1} \frac{m(1-\lambda)}{\lambda} =$
$\frac{m+1}{\lambda z} DR_\lambda^{m+1} f(z,\zeta) - \left(m-1+\frac{1}{\lambda}\right)\left(DR_\lambda^m f(z,\zeta)\right)'_z - \frac{m(1-\lambda)}{\lambda z} DR_\lambda^m f(z,\zeta) =$
$\frac{m+1}{\lambda z} DR_\lambda^{m+1} f(z,\zeta) - \left(m-1+\frac{1}{\lambda}\right) p(z,\zeta) - \frac{m(1-\lambda)}{\lambda z} DR_\lambda^m f(z,\zeta)$.

Therefore $p(z,\zeta) + \frac{\lambda}{m\lambda+1} zp'_z(z,\zeta) = \frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z,\zeta)$.

We have $p(z,\zeta) + \frac{\lambda}{m\lambda+1} zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + \frac{n\lambda}{m\lambda+1} zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 2 we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\left(DR_\lambda^m f(z,\zeta)\right)'_z \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and this result is sharp.

Theorem No. 7 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta) = 1$. If $\lambda \geq 0$, $m,n \in N$, $f \in \mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \zeta \in \overline{U},$$

holds, then

$$\left(DR_{\lambda}^{m}f\left(z,\zeta\right)\right)_{z}' \prec\prec g\left(z,\zeta\right) \prec\prec h\left(z,\zeta\right), z\in U, \zeta\in\overline{U},$$

where $g\left(z,\zeta\right)=\frac{m\lambda+1}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}}\int_{0}^{z}h\left(t,\zeta\right)t^{\frac{m\lambda+1}{\lambda n}-1}dt$ *is convex and it is the best dominant.*

Proof. With notation
$p\left(z,\zeta\right)=\left(DR_{\lambda}^{m}f\left(z,\zeta\right)\right)_{z}'=1+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}\left[1+\left(j-1\right)\lambda\right]^{m}a_{j}^{2}\left(\zeta\right)z^{j-1}$ and $p\left(0,\zeta\right)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_{j}\left(\zeta\right)z^{j}$,
$p\left(z,\zeta\right)+\frac{\lambda}{m\lambda+1}zp_{z}'\left(z,\zeta\right)=\frac{m+1}{(m\lambda+1)z}DR_{\lambda}^{m+1}f\left(z,\zeta\right)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_{\lambda}^{m}f\left(z,\zeta\right).$

We have $p\left(z,\zeta\right)+\frac{\lambda}{m\lambda+1}zp'\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$. Since $p\left(z,\zeta\right)\in\mathcal{H}^{*}\left[1,n,\zeta\right]$, using Lemma 1 for $\gamma=\frac{m\lambda+1}{\lambda}$, we obtain $p\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\left(DR_{\lambda}^{m}f\left(z,\zeta\right)\right)'\prec\prec g\left(z,\zeta\right)=\frac{m\lambda+1}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}}\int_{0}^{z}h\left(t,\zeta\right)t^{\frac{m\lambda+1}{\lambda n}-1}dt\prec\prec h\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, and $g\left(z,\zeta\right)$ is convex and it is the best dominant.

Corollary No. 2 *Let* $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ *a convex function in* $U\times\overline{U}$, $0\le\beta<1$. *If* $\lambda\ge0$, $m,n\in N$, $f\in\mathcal{A}_{n\zeta}^{*}$ *and verifies the strong differential subordination*

$$\frac{m+1}{\left(m\lambda+1\right)z}DR_{\lambda}^{m+1}f\left(z,\zeta\right)-\frac{m\left(1-\lambda\right)}{\left(m\lambda+1\right)z}DR_{\lambda}^{m}f\left(z,\zeta\right)\prec\prec h(z,\zeta),\quad z\in U,\zeta\in\overline{U}, \tag{14}$$

then

$$\left(DR_{\lambda}^{m}f\left(z,\zeta\right)\right)_{z}'\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right), z\in U,\zeta\in\overline{U},$$

where g *is given by* $g(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)(m\lambda+1)}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}}\int_{0}^{z}\frac{t^{\frac{m\lambda+1}{\lambda n}-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. *The function* g *is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 7 and considering $p(z,\zeta)=\left(DR_{\lambda}^{m}f(z,\zeta)\right)_{z}'$, the strong differential subordination (14) becomes

$$p(z,\zeta)+\frac{\lambda}{m\lambda+1}zp_{z}'(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z},\quad z\in U,\zeta\in\overline{U}.$$

By using Lemma 1 for $\gamma=\frac{m\lambda+1}{\lambda}$, we have $p\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$\left(DR_{\lambda}^{m}f(z,\zeta)\right)_{z}'\prec\prec g\left(z,\zeta\right)=\frac{m\lambda+1}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}}\int_{0}^{z}h\left(t,\zeta\right)t^{\frac{m\lambda+1}{\lambda n}-1}dt=$$

$$\frac{m\lambda+1}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}}\int_{0}^{z}t^{\frac{m\lambda+1}{\lambda n}-1}\frac{\zeta+\left(2\beta-\zeta\right)t}{1+t}dt=2\beta-\zeta+\frac{2(\zeta-\beta)\left(m\lambda+1\right)}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}}\int_{0}^{z}\frac{t^{\frac{m\lambda+1}{\lambda n}-1}}{1+t}dt,$$

$z \in U$, $\zeta \in \overline{U}$.

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