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TURÁN TYPE INEQUALITIES FOR SOME $(q,k)-$ SPECIAL FUNCTIONS

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Abstract. The aim of this paper is to establish new Turán-type inequalities involving the $(q,k)-$polygamma functions.As an application, when $q \to 1$ and $k \to 1$, we obtain results from [12] and [13].

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1. Introduction

The inequalities of the type

$$f_n(x)f_{n+2}(x) - f_{n+1}^2(x) \leq 0$$

have many applications in pure mathematics as in other branches of science. They are named by Karlin and Szegő [4], Turán-type inequalities because the first of these type of inequalities was introduced by Turán [15]. More precisely, he used some results of Szegő [14] to prove the previous inequality for $x \in (-1,1)$, where f_n is the Legendre polynomial of degree n. This classical result has been extended in many directions, as ultraspherical polynomials, Laguerre and Hermite polynomials, or Bessel functions, and so forth. Many results of Turán-type have been established on the zeros of special functions.

Recently, W. T. Sulaiman [13] proved some Turán-type inequalities for some q-special functions as well as the polygamma functions, by using the following inequality:

Let $a \in R_+ \cup \{\infty\}$ and let f and g be two nonnegative functions. Then

$$\Big(\int_0^a g(x)f^{\frac{m+n}{2}}d_qx\Big)^2 \leq \Big(\int_0^a g(x)f^m d_qx\Big)\Big(\int_0^a g(x)f^n d_qx\Big) \tag{1}$$

Lets give some definitions for gamma and polygamma function.

The Euler gamma function $\Gamma(x)$ is defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [2]):

$$\psi(x)=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}dt=-\gamma-\frac{1}{x}+\sum_{n\geq1}\frac{x}{n(n+x)}, \tag{2}$$

where $\gamma=0.57721\cdots$ denotes Euler's constant.

Jackson defined the q-analogue of the gamma function as

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0<q<1, \tag{3}$$

and

$$\Gamma_q(x)=\frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}},\ q>1, \tag{4}$$

where $(a;q)_\infty=\prod_{j\geq0}(1-aq^j)$.

The q-gamma function has the following integral representation

$$\Gamma_q(t)=\int_0^\infty x^{t-1}E_q^{-qx}d_qx,$$

where $E_q^x=\sum_{j=0}^\infty q^{\frac{j(j-1)}{2}}\frac{x^j}{[j]!}=(1+(1-q)x)_q^\infty$, which is the q-analogue of the classical exponential function. The q-analogue of the psi function is defined for $0<q<1$ as the logarithmic derivative of the q-gamma function, that is, $\psi_q(x)=\frac{d}{dx}\log\Gamma_q(x)$. It is well known that $\Gamma_q(x)\to\Gamma(x)$ and $\psi_q(x)\to\psi(x)$ as $q\to1^-$. From (3), for $0<q<1$ and $x>0$ we get

$$\psi_q(x)=-\log(1-q)+\log q\sum_{n\geq0}\frac{q^{n+x}}{1-q^{n+x}}=-\log(1-q)+\log q\sum_{n\geq1}\frac{q^{nx}}{1-q^n}$$

and from (4) for $q>1$ and $x>0$ we obtain

$$\begin{aligned}\psi_q(x)&=-\log(q-1)+\log q\left(x-\tfrac{1}{2}-\sum_{n\geq0}\tfrac{q^{-n-x}}{1-q^{-n-x}}\right)\\&=-\log(q-1)+\log q\left(x-\tfrac{1}{2}-\sum_{n\geq1}\tfrac{q^{-nx}}{1-q^{-n}}\right).\end{aligned}$$

If $q \in (0, 1)$, using the second representation of $\psi_q(x)$ given in () can be shown that

$$\psi_q^{(k)}(x) = \log^{k+1} q \sum_{n \geq 1} \frac{n^k \cdot q^{nx}}{1 - q^n}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 1$, for all $k \geq 1$. If $q > 1$, from the second representation of $\psi_q(x)$ given in () we obtain

$$\psi'_q(x) = \log q \Big(1 + \sum_{n \geq 1} \frac{nq^{-nx}}{1 - q^{-nx}}\Big)$$

and for $k \geq 2$,

$$\psi_q^{(k)}(x) = (-1)^{k-1} \log^{k+1} q \sum_{n \geq 1} \frac{n^k q^{-nx}}{1 - q^{-nx}}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 0$, for all $q > 1$.

Definition 1.1. *Let $x \in C, k \in R$ and $n \in N^+$, the Pochhammer $k-$symbol is given by*

$$(x)_{n,k} = x(x + k)(x + 2k) \cdots (x + (n - 1)k). \tag{5}$$

Definition 1.1. *For $k > 0$, the $k-$gamma function Γ_k is given by*

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n! k^n (nk)^{\frac{x}{k} - 1}}{(x)_{n,k}}, x \in C \setminus kZ^- \tag{6}$$

For $x \in C,\ Re(x) > 0$, the function Γ_k is given by the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt. \tag{7}$$

k-analogue of the psi function is defined as the logarithmic derivative of the Γ_k function, that is

$$\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x) = \frac{\Gamma'_k(x)}{\Gamma_k(x)}, k > 0. \tag{8}$$

The function $\psi_k(x)$ defined by (8) has the following series representation

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(x + nk)} \tag{9}$$

$$\psi_k^{(n)}(x) = (-1)^{n+1} \cdot n! \sum_{p=0}^{\infty} \frac{1}{(x + pk)^{n+1}} \tag{10}$$

Rafael Díaz (see [3]) defined the $(q,k)-$analogue of the gamma function as

$$\Gamma_{q,k} = \frac{(1-q^k)^{\infty}_{q,k}}{(1-q^k)^{\infty}_{q,k} \cdot (1-q^k)^{\frac{x}{k}-1}} \tag{11}$$

where $(x+y)^n_{q,k} = \prod_{j=0}^{n-1}(x+q^{jk}y)$.

We define the $(q,k)-$ analogue of the psi function, for $0<q<1$ and $k>0$, as the logarithmic derivative of the $(q,k)-$ gamma function, that is, $\psi_{q,k}(x) = \frac{d}{dx}\ln\Gamma_{q,k}(x)$. Many properties of the $(q,k)-$gamma function were derived by Díaz [4]. It is well known that $\Gamma_{q,k}(x) \to \Gamma_q(x)$ as $k \to 1$. From (11), for $0<q<1$ and $x>0$ we get

$$\psi_{q,k}(x) = \frac{-\log((1-q)}{k} + \log q \sum_{n\geq 1} \frac{q^{nkx}}{1-q^{nk}} \tag{12}$$

One can easily show that $\psi_{(q,k)}(x) \to \psi_q(x)$ as $k \to 1$. If $q \in (0,1)$ then by using the second representation of $\psi_{q,k}(x)$ given in (12) can be shown that

$$\psi^{(j)}_{(q,k)}(x) = \log^{j+1} q \sum_{n\geq 1} \frac{n^j k^j \cdot q^{nkx}}{1-q^{nk}} \tag{13}$$

2. Main Results

Theorem 2.1. For $n = 1,2,3,\ldots$, let $\psi_{(q,k),n} = \psi^{(n)}_{(q,k)}$ the n-th derivative of the function $\psi_{(q,k)}$. Then

$$\psi_{(q,k),\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{t}\right) \leq \psi^{\frac{1}{s}}_{(q,k),m}(x)\psi^{\frac{1}{l}}_{(q,k),n}(y), \tag{14}$$

where $\frac{m+n}{2}$ is an integer, $s>1$, $\frac{1}{s}+\frac{1}{l}=1$.

Proof. Let m and n be two integers of the same parity. From (13), it follows that:

$$\begin{aligned}\psi_{(q,k),\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{t}\right) &= \log^{\frac{m}{s}+\frac{n}{l}+1} q \sum_{i\geq 1} \frac{i^{\frac{m}{s}+\frac{n}{l}} k^{\frac{m}{s}+\frac{n}{l}} \cdot q^{ik\left(\frac{x}{s}+\frac{y}{t}\right)}}{1-q^{ik}} \\ &= \log^{\frac{m+1}{s}+\frac{n+1}{l}} q \sum_{i\geq 1} \frac{i^{\frac{m}{s}} k^{\frac{m}{s}} \cdot q^{\frac{ikx}{s}} i^{\frac{n}{l}} k^{\frac{n}{l}} \cdot q^{\frac{iky}{l}}}{\left(1-q^{ik}\right)^{\frac{1}{s}} \cdot \left(1-q^{ik}\right)^{\frac{1}{l}}} \\ &\leq \left(\log^{m+1} q \sum_{i\geq 1} \frac{i^m k^m \cdot q^{ikx}}{1-q^{ik}}\right)^{\frac{1}{s}} \cdot \left(\log^{n+1} q \sum_{i\geq 1} \frac{i^n k^n \cdot q^{iky}}{1-q^{ik}}\right)^{\frac{1}{l}} \\ &= \psi^{\frac{1}{s}}_{(q,k),m}(x)\psi^{\frac{1}{l}}_{(q,k),n}(y)\end{aligned}$$

Remark 2.2. *Let k tend to 1 then we obtain Theorem 2.2 from [13]*

$$\psi_{q,\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{t}\right) \leq \psi^{\frac{1}{s}}_{q,m}(x)\psi^{\frac{1}{l}}_{q,n}(y), \tag{15}$$

On putting $y = x$ and for k, q tend to 1, then we obtain Theorem 2.1 from [12]

$$\psi_{q,\frac{m}{s}+\frac{n}{l}}(x) \leq \psi^{\frac{1}{s}}_{q,m}(x)\psi^{\frac{1}{l}}_{q,n}(y), \tag{16}$$

Another type via Minkowski's inequality is the following. *Theorem 2.3 For $n = 1, 2, 3, \dots,$ let $\psi_{(q,k),n} = \psi^{(n)}_{(q,k)}$ the n-th derivative of the function $\psi_{(q,k)}$. Then*

$$\left(\psi_{(q,k),m}(x) + \psi_{(q,k),n}(y)\right)^{\frac{1}{p}} \leq \psi^{\frac{1}{p}}_{(q,k),m}(x) + \psi^{\frac{1}{p}}_{(q,k),n}(y), \tag{17}$$

where $\frac{m+n}{2}$ is an integer, $p \geq 1$. *Proof.* Since,

$$(a+b)^p \geq a^p + b^p, \quad a, b \geq 0, \quad p \geq 1,$$

$$
\begin{aligned}
&\Big(\psi_{(q,k),m}(x)+\psi_{(q,k),n}(y)\Big)^{\frac{1}{p}}\\
&\quad=\Big[\sum_{i\geq 1}\Big(\log^{m+1}q\frac{i^m k^m\cdot q^{ikx}}{1-q^{ik}}+\log^{n+1}q\frac{i^n k^n\cdot q^{ikx}}{1-q^{ik}}\Big)\Big]^{\frac{1}{p}}\\
&\quad=\Big[\sum_{i\geq 1}\Big(\Big(\log^{\frac{m+1}{p}}q\frac{i^{\frac{m}{p}}k^{\frac{m}{p}}\cdot q^{\frac{ikx}{p}}}{(1-q^{ik})^{\frac{1}{p}}}\Big)^p+\Big(\log^{\frac{n+1}{p}}q\frac{i^{\frac{n}{p}}k^{\frac{n}{p}}\cdot q^{\frac{iky}{p}}}{(1-q^{ik})^{\frac{1}{p}}}\Big)^p\Big)\Big]^{\frac{1}{p}}\\
&\quad\leq\Big[\sum_{i\geq 1}\Big(\Big(\log^{\frac{m+1}{p}}q\frac{i^{\frac{m}{p}}k^{\frac{m}{p}}\cdot q^{\frac{ikx}{p}}}{(1-q^{ik})^{\frac{1}{p}}}\Big)^p\Big]^{\frac{1}{p}}+\Big[\sum_{i\geq 1}\Big(\log^{\frac{n+1}{p}}q\frac{i^{\frac{n}{p}}k^{\frac{n}{p}}\cdot q^{\frac{iky}{p}}}{(1-q^{ik})^{\frac{1}{p}}}\Big)^p\Big)\Big]^{\frac{1}{p}}\\
&\quad=\Big[\log^{m+1}q\sum_{i\geq 1}\frac{i^m k^m\cdot q^{ikx}}{1-q^{ik}}\Big]^{\frac{1}{p}}+\Big[\log^{n+1}q\sum_{i\geq 1}\frac{i^n k^n\cdot q^{iky}}{1-q^{ik}}\Big]^{\frac{1}{p}}\\
&\quad=\psi^{\frac{1}{p}}_{(q,k),m}(x)+\psi^{\frac{1}{p}}_{(q,k),n}(y)
\end{aligned}
$$

Remark 2.3. Let k,q tend to 1then we have

$$
\Big(\psi_m(x)+\psi_n(y)\Big)^{\frac{1}{p}}\leq\psi_m^{\frac{1}{p}}(x)+\psi_n^{\frac{1}{p}}(y), \tag{18}
$$

Theorem 2.4. For every $x>0$ and integers $n\geq 1$, we have:

1. *If n is odd, then* $\Big(\exp\psi^{(n)}_{(q,k)}(x)\Big)^2\geq\exp\psi^{(n+1)}_{(q,k)}(x)\cdot\exp\psi^{(n-1)}_{(q,k)}(x)$
2. *If n is even, then* $\Big(\exp\psi^{(n)}_{(q,k)}(x)\Big)^2\leq\exp\psi^{(n+1)}_{(q,k)}(x)\cdot\exp\psi^{(n-1)}_{(q,k)}(x)$

Proof. We use (13) to estimate the expression

$$\psi^{(n)}_{(q,k)}(x) - \frac{\psi^{(n+1)}_{(q,k)}(x) + \psi^{(n-1)}_{(q,k)}(x)}{2} =$$

$$\log^{n+1} q \sum_{i\geq 1} \frac{i^n k^n \cdot q^{ikx}}{1-q^{ik}}$$

$$- \frac{\log^{n+2} q \sum_{i\geq 1} \frac{i^{n+1}k^{n+1}\cdot q^{ikx}}{1-q^{ik}} + \log^n q \sum_{i\geq 1} \frac{i^{n-1}k^{n-1}\cdot q^{ikx}}{1-q^{ik}}}{2}$$

$$= \log^n q \sum_{i\geq 1} \frac{i^{n-1}k^{n-1} \cdot q^{ikx}}{1-q^{ik}} \Big(ik \log q - \frac{i^2k^2 \log^2 q + 1}{2} \Big)$$

$$= -\log^n q \sum_{i\geq 1} \frac{i^{n-1}k^{n-1} \cdot q^{ikx}}{1-q^{ik}} \frac{(ik\log q - 1)^2}{2}$$

Now, the conclusion follows by exponentiating the inequality

$$\psi^{(n)}_{(q,k)}(x) \geq (\leq) \frac{\psi^{(n+1)}_{(q,k)}(x) + \psi^{(n-1)}_{(q,k)}(x)}{2}$$

as n is odd, respective even.

Remark 2.5.
Let q, k tend to 1 *then we obtain generalization of Theorem 3.3 from [12]*

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