Acta Universitatis Apulensis No. 34/2013
ISSN: 1582-5329 pp. 265-273

$\mathcal{S}-\alpha$ SURFACES OF TIMELIKE BIHARMONIC $\mathcal{S}$-CURVES ACCORDING TO SABBAN FRAME IN $\mathbb{H}$

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Abstract. In this paper, we study $\mathcal{S}-\alpha$ surfaces according to Sabban frame in the Lorentzian Heisenberg group $\mathbb{H}$. We characterize the timelike biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Lorentzian Heisenberg group $\mathbb{H}$. Finally, we find explicit parametric equations of $\mathcal{S}-\alpha$ surfaces according to Sabban Frame.

2000 *Mathematics Subject Classification*: 31B30,58E20.

1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \operatorname{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \operatorname{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we study $\mathcal{S}-\alpha$ surfaces according to Sabban frame in the Lorentzian Heisenberg group $\mathbb{H}$. Secondly, we characterize the timelike biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Heisenberg group $\mathbb{H}$. Finally, we find explicit parametric equations of $\mathcal{S}-\alpha$ surfaces according to Sabban Frame.

2. Preliminaries

Heisenberg group $\mathbb{H}$ can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y})$$

$\mathbb{H}$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group. The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on $\mathbb{H}$ is

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1=\frac{\partial}{\partial z},\ \mathbf{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\ \mathbf{e}_3=\frac{\partial}{\partial x}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=1,\quad g(\mathbf{e}_3,\mathbf{e}_3)=-1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla=\frac{1}{2}\begin{pmatrix}0 & \mathbf{e}_3 & \mathbf{e}_2\\ \mathbf{e}_3 & 0 & \mathbf{e}_1\\ \mathbf{e}_2 & -\mathbf{e}_1 & 0\end{pmatrix}, \tag{2.2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k=1,2,3\}.$$

The unit pseudo-Heisenberg sphere (Lorentzian Heisenberg sphere) is defined by

$$\left(\mathbb{S}_1^2\right)_{\mathbb{H}}=\left\{\boldsymbol{\beta}\in\mathbb{H}:g\left(\boldsymbol{\beta},\boldsymbol{\beta}\right)=1\right\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = -g(R(X,Y)Z,W).$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{121} = \frac{1}{4}\mathbf{e}_2, \quad R_{131} = \frac{1}{4}\mathbf{e}_3, \quad R_{232} = -\frac{3}{4}\mathbf{e}_3$$

and

$$R_{1212} = -\frac{1}{4}, \quad R_{1313} = \frac{1}{4}, \quad R_{2323} = -\frac{3}{4}. \tag{2.3}$$

3. Timelike Biharmonic S-Curves According To Sabban Frame

Let $\gamma : I \longrightarrow \mathbb{H}$ be a timelike curve in the Lorentzian Heisenberg group $\mathbb{H}$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}$ along γ defined as follows: $\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, 3.1 \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{1}$$

where κ is the curvature of γ and τ is its torsion,

$$\begin{aligned} g(\mathbf{T},\mathbf{T}) &= -1, \; g(\mathbf{N},\mathbf{N}) = 1, \; g(\mathbf{B},\mathbf{B}) = 1, \\ g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned}$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \longrightarrow \left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ be unit speed spherical timelike curve. We denote σ as the arc-length parameter of α . Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along α. This frame is called the Sabban frame of α on $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$. Then we have the following spherical Frenet-Serret formulae of α :

$$\begin{aligned} \nabla_{\mathbf{t}}\alpha &= \mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{t} &= \alpha + \kappa_g\mathbf{s}, 3.2 \\ \nabla_{\mathbf{t}}\mathbf{s} &= \kappa_g\mathbf{t}, \end{aligned} \tag{2}$$

where κ_g is the geodesic curvature of the timelike curve α on the $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= -1,\ g(\alpha,\alpha)=1,\ g(\mathbf{s},\mathbf{s})=1, \\ g(\mathbf{t},\alpha) &= g(\mathbf{t},\mathbf{s})=g(\alpha,\mathbf{s})=0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned} \alpha &= \alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2+\alpha_3\mathbf{e}_3, \\ \mathbf{t} &= t_1\mathbf{e}_1+t_2\mathbf{e}_2+t_3\mathbf{e}_3, 3.3 \\ \mathbf{s} &= s_1\mathbf{e}_1+s_2\mathbf{e}_2+s_3\mathbf{e}_3. \end{aligned} \tag{3}$$

To separate a biharmonic curve according to Sabban frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathcal{S}$-curve.

Theorem 3.1. *$\alpha : I \longrightarrow \left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ is a timelike biharmonic $\mathcal{S}$-curve if and only if*

$$\begin{aligned} \kappa_g &= \text{constant} \neq 0, \\ 1+\kappa_g^2 &= [-\frac{1}{4}+\frac{1}{2}s_1^2]+\kappa_g[\alpha_1 s_1], 3.4 \\ \kappa_g^3 &= \alpha_3 s_3-\kappa_g[\frac{1}{4}-\frac{1}{2}\alpha_1^2]. \end{aligned} \tag{4}$$

4. $\mathcal{S}-\alpha$ Surfaces Of Timelike Biharmonic $\mathcal{S}$-Curves According To Sabban Frame

To separate a α surface according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathcal{S}-\alpha$ surface. The purpose of this section is to study $\mathcal{S}-\alpha$ surfaces of timelike biharmonic $\mathcal{S}$-curve in the Lorentzian Heisenberg group Heis^3. The $\mathcal{S}-\alpha$ surface of γ is a ruled surface

$$\mathcal{E}^{\mathcal{S}}(\sigma,u)=\alpha(\sigma)+u\alpha(\sigma). \tag{4.1}$$

Theorem 4.1. *Let $\mathcal{E}^{\mathcal{S}}$ be a $\mathcal{S}-\alpha$ surface of a unit speed non-geodesic timelike biharmonic $\mathcal{S}-$curve in the Heisenberg group Heis^3. Then, the parametric equations* *of* $\mathcal{S}-\alpha$ *surface of* α *are*

$$\begin{aligned} \boldsymbol{x}_{\mathcal{E}^{\mathcal{S}}}(\sigma,u) &= (1+u)\,[\frac{\cosh\mathcal{A}}{\mathcal{B}_0}\sinh[\mathcal{B}_0\sigma+\mathcal{B}_1]+\mathcal{B}_2], \\ \boldsymbol{y}_{\mathcal{E}^{\mathcal{S}}}(\sigma,u) &= (1+u)\,[\frac{\cosh\mathcal{A}}{\mathcal{B}_0}\cosh[\mathcal{B}_0\sigma+\mathcal{B}_1]+\mathcal{B}_3], 4.2 \qquad (5) \\ \boldsymbol{z}_{\mathcal{E}^{\mathcal{S}}}(\sigma,u) &= (1+u)\,[\sinh\mathcal{A}\sigma+\frac{\cosh^2\mathcal{A}}{2\mathcal{B}_0^2}[\mathcal{B}_0\sigma+\mathcal{B}_1]-\frac{\cosh^2\mathcal{A}}{4\mathcal{B}_0^2}\sinh 2[\mathcal{B}_0\sigma+\mathcal{B}_1] \\ &\quad -\frac{\mathcal{B}_2}{\mathcal{B}_0}\cosh\mathcal{A}\cosh[\mathcal{B}_0\sigma+\mathcal{B}_1]+u\sinh\mathcal{A} \\ &\quad +u\cosh\mathcal{A}(\frac{\cosh\mathcal{A}}{\mathcal{B}_0}\sinh[\mathcal{B}_0\sigma+\mathcal{B}_1]+\mathcal{B}_2)\sinh[\mathcal{B}_0\sigma+\mathcal{B}_1]+\mathcal{B}_4], \end{aligned}$$

where $\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4$ *are constants of integration and*

$$\mathcal{B}_0=\frac{\sqrt{1+\kappa_g^2}}{\cosh\mathcal{A}}-\sinh\mathcal{A}.$$

Proof. By the Sabban formula, we have the following equation

$$\mathbf{t}=\sinh\mathcal{A}\mathbf{e}_1+\cosh\mathcal{A}\sinh[\mathcal{B}_0\sigma+\mathcal{B}_1]\mathbf{e}_2+\cosh\mathcal{A}\cosh[\mathcal{B}_0\sigma+\mathcal{B}_1]\mathbf{e}_3. \qquad (4.3)$$

Using (2.1) in (4.3), we obtain

$$\begin{aligned} \mathbf{t} &= (\cosh\mathcal{A}\cosh[\mathcal{B}_0\sigma+\mathcal{B}_1],\cosh\mathcal{A}\sinh[\mathcal{B}_0\sigma+\mathcal{B}_1], 4.4 \qquad (6) \\ &\quad \sinh\mathcal{A}+\cosh\mathcal{A}(\frac{\cosh\mathcal{A}}{\mathcal{B}_0}\sinh[\mathcal{B}_0\sigma+\mathcal{B}_1]+\mathcal{B}_2)\sinh[\mathcal{B}_0\sigma+\mathcal{B}_1]), \end{aligned}$$

where $\mathcal{B}_1,\mathcal{B}_2$ are constants of integration.

Consequently, the parametric equations of $\mathcal{E}^{\mathcal{S}}$ can be found from (4.1), (4.4). This concludes the proof of Theorem.

Thus, the following result is valid.

Lemma 4.2. *Let* $\alpha: I\longrightarrow\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ *be a unit speed non-geodesic timelike bihar-*

monic $\mathcal{S}$-curve. *Then, the position vector of* α *is*

$$\begin{aligned} \alpha(\sigma) &= [\sinh \mathcal{A}\sigma + \frac{\cosh^2 \mathcal{A}}{2\mathcal{B}_0^2}[\mathcal{B}_0\sigma + \mathcal{B}_1] - \frac{\cosh^2 \mathcal{A}}{4\mathcal{B}_0^2}\sinh 2[\mathcal{B}_0\sigma + \mathcal{B}_1] \\ &-\frac{\mathcal{B}_2}{\mathcal{B}_0}\cosh \mathcal{A}\cosh[\mathcal{B}_0\sigma + \mathcal{B}_1] + [\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\cosh[\mathcal{B}_0\sigma + \mathcal{B}_1] \\ &+\mathcal{B}_3][\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\sinh[\mathcal{B}_0\sigma + \mathcal{B}_1] + \mathcal{B}_2] + \mathcal{B}_4]\mathbf{e}_1 4.5 \\ &+[\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\cosh[\mathcal{B}_0\sigma + \mathcal{B}_1] + \mathcal{B}_3]\mathbf{e}_2 \\ &+[\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\sinh[\mathcal{B}_0\sigma + \mathcal{B}_1] + \mathcal{B}_2]\mathbf{e}_3, \end{aligned} \tag{7}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ *are constants of integration and*

$$\mathcal{B}_0 = \frac{\sqrt{1+\kappa_g^2}}{\cosh \mathcal{A}} - \sinh \mathcal{A}.$$

Summarizing up the above, we have

Theorem 4.3. *Let* $\mathcal{E}^{\mathcal{S}}$ *be a* $\mathcal{S}-\alpha$ *surface of a unit speed non-geodesic timelike biharmonic* $\mathcal{S}-$*curve in the Lorentzian Heisenberg group* $Heis^3$. *Then, the equation of* $\mathcal{S}-\alpha$ *surface of* α *is*

$$\begin{aligned} \mathcal{E}^{\mathcal{S}}(\sigma, u) &= (1+u)[\sinh \mathcal{A}\sigma + \frac{\cosh^2 \mathcal{A}}{2\mathcal{B}_0^2}[\mathcal{B}_0\sigma + \mathcal{B}_1] - \frac{\cosh^2 \mathcal{A}}{4\mathcal{B}_0^2}\sinh 2[\mathcal{B}_0\sigma + \mathcal{B}_1] \\ &-\frac{\mathcal{B}_2}{\mathcal{B}_0}\cosh \mathcal{A}\cosh[\mathcal{B}_0\sigma + \mathcal{B}_1] + [\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\cosh[\mathcal{B}_0\sigma + \mathcal{B}_1] \\ &+\mathcal{B}_3][\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\sinh[\mathcal{B}_0\sigma + \mathcal{B}_1] + \mathcal{B}_2] + \mathcal{B}_4]\mathbf{e}_1 4.6 \\ &+(1+u)[\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\cosh[\mathcal{B}_0\sigma + \mathcal{B}_1] + \mathcal{B}_3]\mathbf{e}_2 \\ &+(1+u)[\frac{\cosh \mathcal{A}}{\mathcal{B}_0}\sinh[\mathcal{B}_0\sigma + \mathcal{B}_1] + \mathcal{B}_2]\mathbf{e}_3, \end{aligned} \tag{8}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ *are constants of integration and*

$$\mathcal{B}_0 = \frac{\sqrt{1+\kappa_g^2}}{\cosh \mathcal{A}} - \sinh \mathcal{A}.$$

Proof. We assume that α is a unit speed timelike biharmonic $\mathcal{S}$-curve. Substituting (2.1) to (4.2), we have (4.6). Thus, the proof is completed.

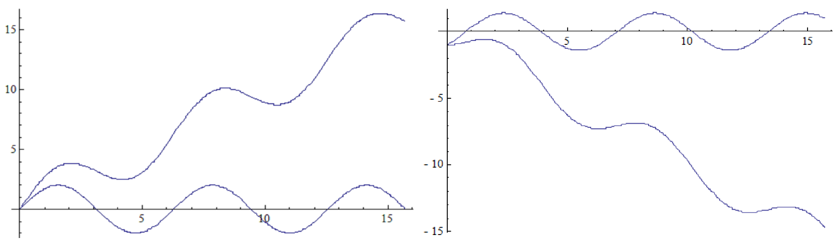

Figure 1:

5. Conclusion

The surfaces obtained by bending these materials can be flattened onto a plane without stretching or tearing. More precisely, there exists a transformation that maps the surface onto the plane, after which the length of any curve drawn on the surface remains the same. Such surfaces, when sufficiently regular, are well known to mathematicians as developable surfaces. Afterwards, we give some results for $\mathcal{S}-\alpha$ surfaces according to Sabban frame in the Lorentzian Heisenberg group Heis^3. We characterize the timelike biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Lorentzian Heisenberg group Heis^3. Finally, we find explicit parametric equations of $\mathcal{S}-\alpha$ surfaces according to Sabban Frame and draw its picture by using Mathematica computer program.

If we use Mathematica in Theorem 4.1 for different constant, yields Figure 1.

References

[1] V. Asil: *Velocities of dual homothetic exponential motions in D^3,* Iranian Journal of Science & Tecnology Transaction A: Science 31 (4) (2007), 265-271.

[2] V. Asil, T. Körpınar: *Construction of Inextensible Flows of Dual Normal Surfaces in the Dual Space D^3*, Bol. Soc. Paran. Mat., 31 (2) (2013), 31-37.

[3] M. Babaarslan and Y. Yayli: *On spacelike constant slope surfaces and Bertrand curves in Minkowski 3-space,* arXiv:1112.1504v2.

[4] L. R. Bishop: *There is More Than One Way to Frame a Curve,* Amer. Math. Monthly 82 (3) (1975), 246-251.

[5] M. Ergüt, E. Turhan and T. Körpınar: *Characterization Of Inextensible Flows Of Spacelike Curves With Sabban Frame In $\mathbb{S}^2_1$*,. Soc. Paran. Mat., 31 (2) (2013), 47-53.

[6] J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

[7] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

[8] T. Körpınar, E. Turhan: *New Special Type Developable Surfaces in terms of Focal Curve of Biharmonic Curve in the Heisenberg Group Heis*^{3}, Middle East Journal of Scientific Research, 7 (4) (2011), 473-477.

[9] T. Körpınar and E. Turhan: *Biharmonic S-Curves According to Sabban Frame in Heisenberg Group Heis*^{3}, Bol. Soc. Paran. Mat. 31 (1) (2013), 205–211.

[10] T. Körpınar, E. Turhan: *Inextensible flows of S-s surfaces of biharmonic S -curves according to Sabban frame in Heisenberg Group Heis*^{3}*,* Lat. Am. J. Phys. Educ. 6 (2) (2012), 250-255.

[11] T. Körpınar, E. Turhan, V. Asil: *Involute Curves Of Timelike Biharmonic Reeb Curves (LCS)*$_3$ *- Manifolds,* Electronic Journal of Theoretical Physics, 9 (26) (2012), 183 – 190.

[12] S. Rahmani: *Metriqus de Lorentz sur les groupes de Lie unimodulaires de dimension trois,* Journal of Geometry and Physics 9 (1992), 295-302.

[13] E. Turhan and T. Körpınar: *Parametric equations of general helices in the sol space* $\mathfrak{Sol}^3$, Bol. Soc. Paran. Mat. 31 (1) (2013), 99–104.

[14] E. Turhan and T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis*^{3}, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

[15] E. Turhan and T. Körpınar: *On Characterization Canal Surfaces around Timelike Horizontal Biharmonic Curves in Lorentzian Heisenberg Group Heis*^{3}, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 66a (2011), 441-449.

[16] E. Turhan, T. Körpınar, *On Smarandache ts Curves of Biharmonic S-Curves According to Sabban Frame in Heisenberg Group Heis*^{3}, Advanced Modeling and Optimization, 14 (2) (2012), 344-349.

[17] E. Turhan, T. Körpınar, *Position Vector Of Spacelike Biharmonic Curves In The Lorentzian Heisenberg Group Heis*^{3}, An. Şt. Univ. Ovidius Constanta, 19 (1) (2011) 285-296.

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