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ON A CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS BASED ON AL-OBOUDI OPERATOR

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Abstract. In this paper we introduce the subclass $TS_p^\lambda(\mu,\alpha,\beta,\delta)$, $0\le\mu<1$, $0\le\alpha<1$, $\lambda,\beta\ge 0$ of analytic functions with negative coefficients. This class is motivated by the study of Sudharsan et al. (2010). We obtain a coefficient characterization, growth and distortion theorems, closure theorem and a convolution result for functions in this class.

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1. Introduction

Let H be the set of functions regular in the unit disc $\Delta=\{z:|z|<1\}$.

Let $A=\{f(z)\in H|f(0)=f'(0)-1=0\}$ and $S=\{f(z)\in A: f(z)$ is univalent in $\Delta\}$ where

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

For $0\le\alpha<1$, let $S^*(\alpha)$ and $K(\alpha)$ denote the subfamilies of S consisting of functions starlike of order α and convex of order α respectively. A function $f\in S$ is called convex of order α $(0\le\alpha<1)$ if and only if $R\left(1+\frac{zf''(z)}{f'(z)}\right)\ge\alpha$ $(z\in\Delta)$. A function $f\in S$ is called starlike of order α $(0\le\alpha<1)$ if and only if $R\left(\frac{zf'(z)}{f(z)}\right)\ge\alpha$.

The subfamily T of S consists of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n, \tag{2}$$

$a_n\ge 0$, for $n=2,3,\ldots,$ $z\in\Delta$. Silverman [8] investigated functions in the classes $T^*(\alpha)=T\cap S^*(\alpha)$ and $C(\alpha)=T\cap K(\alpha)$.

Let $n \in N_0$ and $\lambda \geq 0$. Denote by D_λ^n, the Al-Oboudi operator [3] defined by $D_\lambda^n : A \to A$,

$$D_\lambda^0 f(z) = f(z)$$
$$D_\lambda^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z)$$
$$D_\lambda^n f(z) = D_\lambda[D_\lambda^{n-1} f(z)].$$

Note that for $f(z)$ given by (1),

$$D_\lambda^n = z + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^n a_j z^j.$$

When $\lambda = 1$, D_λ^n is the Sălăgean differential operator [6], $D^n : A \to A$, $n \in N$ defined as

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^n f(z) = D[D^{n-1} f(z)].$$

Extending the Al-Oboudi operator D_λ^n Acu and Owa [2] considered the operator D_λ^β which is defined as follows:

Definition 1.1. **[2]** Let $\beta, \lambda \in R$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined by

$$D_\lambda^\beta : A \to A, \quad D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j z^j$$

Remark 1.1. If $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \ldots$, $\beta \geq 0$, $\lambda \geq 0$, $z \in \Delta$, then

$$D_\lambda^\beta f(z) = z - \sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j z^j$$

Definition 1.2. **[1]** Let $f(z) \in T$ be of the form given by (2). Then for $\alpha \in [0, 1)$, $\lambda \geq 0$, $\beta \geq 0$, $f(z)$ is in the class $TL_\beta(\alpha)$ if $Re\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} > \alpha$, and is in the class $T^C L_\beta(\alpha)$ if $Re\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} > \alpha$.

Using the operator D_λ^β Sudharsan et al. [10] introduced and studied the classes $TS_p^\lambda(\alpha,\beta)$ and $TV^\lambda(\alpha,\beta)$ defined as follows.

Definition 1.3. **[10]** Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \ldots$, $z \in \Delta$. We say that $f(z)$ in in the class $TS_p^\lambda(\alpha,\beta)$ if

$$Re\left\{\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} - \alpha\right\} \geq \left|\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} - 1\right|, \quad \alpha \in [-1,1), \lambda \geq 0, \beta \geq 0.$$

We say that $f(z)$ is in the class $TV^\lambda(\alpha,\beta)$ if

$$Re\left\{\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} - \alpha\right\} \geq \left|\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} - 1\right|, \quad \alpha \in [-1,1), \lambda \geq 0, \beta \geq 0.$$

Motivated by the above definitions, we now define the following subclass $S_p^\lambda(\mu,\alpha,\beta,\delta)$.

Definition 1.4. For $0 \leq \mu < 1$, $0 \leq \alpha < 1$, $\delta, \beta \geq 0$. Let $S_p^\lambda(\mu,\alpha,\beta,\delta)$ be the subclass of S consisting of functions of the form (1) and satisfying the condition

$$Re\left\{\frac{D_\lambda^{\beta+1} f(z)}{(1-\mu)D_\lambda^\beta f(z) + \mu D_\lambda^{\beta+1} f(z)} - \alpha\right\} > \delta\left|\frac{D_\lambda^{\beta+1} f(z)}{(1-\mu)D_\lambda^\beta f(z) + \mu D_\lambda^{\beta+1} f(z)} - 1\right|$$

We further let $TS_p^\lambda(\mu,\alpha,\beta,\delta) = S_p^\lambda(\mu,\alpha,\beta,\delta) \cap T$, $z \in \Delta$.

Remark 1.2.

1. $TS_p^1(\mu,\alpha,0,\delta) = TS_p(\mu,\alpha,\delta)$ [5].
2. $TS_p^1(0,0,0,\delta) = TS_p(\delta)$ [9].
3. $TS_p^1(0,\alpha,0,1) = TS_p(\alpha)$ [4].
4. $TS_p^\lambda(0,\alpha,\beta,1) = TS_p^\lambda(\alpha,\beta)$, for $\alpha \in [0,1)$ [10].
5. $TS_p^\lambda(0,\alpha,\beta,0) = TL_\beta(\alpha)$ [1].
6. $TS_p^1(0,\alpha,0,1) = T^*\left(\frac{1+\alpha}{2}\right)$ [8].

In this paper, we obtain the sharp result for coefficient estimates, distortion theorems, growth theorems and convolution result for the class $TS_p^\lambda(\mu,\alpha,\beta,\delta)$.

2. Main Results

Theorem 2.1. *A function $f(z)$ of the form (2) is in $TS_p^{\lambda}(\mu,\alpha,\beta,\delta)$ if and only if*

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}\left[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}\right]a_j\leq 1-\alpha. \quad (3)$$

Proof. Let $f\in TS_p^{\lambda}(\mu,\alpha,\beta,\delta)$ with $\lambda,\beta\geq 0$, $0\leq\mu<1$, $0\leq\alpha<1$.
We have

$$Re\left\{\frac{D_{\lambda}^{\beta+1}f(z)}{(1-\mu)D_{\lambda}^{\beta}f(z)+\mu D_{\lambda}^{\beta+1}f(z)}-\alpha\right\}>\delta\left|\frac{D_{\lambda}^{\beta+1}f(z)}{(1-\mu)D_{\lambda}^{\beta}f(z)+\mu D_{\lambda}^{\beta+1}f(z)}-1\right|.$$

If we take $z\in[0,1)$, $\lambda,\beta\geq 0$, we have

$$\frac{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_jz^{j-1}}{(1-\mu)\left[1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_jz^{j-1}\right]+\mu\left[\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_jz^{j-1}\right]}-\alpha\geq$$
$$\delta\left[1-\left[\frac{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_jz^{j-1}}{(1-\mu)\left[1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_jz^{j-1}\right]+\mu\left[\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_jz^{j-1}\right]}\right]\right]$$

This yields

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}\left[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}\right]a_jz^{j-1}\leq 1-\alpha.$$

Letting $z\to 1^-$ along the real axis we have,

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}\left[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}\right]a_j\leq 1-\alpha.$$

Conversely, let us take $f(z)\in T$ for which the relation (3) hold.
It suffices to show that

$$\delta\left|\frac{D_{\lambda}^{\beta+1}f(z)}{(1-\mu)D_{\lambda}^{\beta}f(z)+\mu D_{\lambda}^{\beta+1}f(z)}-1\right|-Re\left\{\frac{D_{\lambda}^{\beta+1}f(z)}{(1-\mu)D_{\lambda}^{\beta}f(z)+\mu D_{\lambda}^{\beta+1}f(z)}-1\right\}\leq 1-\alpha,$$

$z \in \Delta$. We have

$$\delta\left|\frac{D_\lambda^{\beta+1}f(z)}{(1-\mu)D_\lambda^\beta f(z)+\mu D_\lambda^{\beta+1}f(z)}-1\right| - Re\left\{\frac{D_\lambda^{\beta+1}f(z)}{(1-\mu)D_\lambda^\beta f(z)+\mu D_\lambda^{\beta+1}f(z)}-1\right\}$$
$$\leq \delta\left|\frac{D_\lambda^{\beta+1}f(z)}{(1-\mu)D_\lambda^\beta f(z)+\mu D_\lambda^{\beta+1}f(z)}-1\right| + \left|\frac{D_\lambda^{\beta+1}f(z)}{(1-\mu)D_\lambda^\beta f(z)+\mu D_\lambda^{\beta+1}f(z)}-1\right|$$
$$\leq (\delta+1)\left|\frac{D_\lambda^{\beta+1}f(z)}{(1-\mu)D_\lambda^\beta f(z)+\mu D_\lambda^{\beta+1}f(z)}-1\right|$$
$$\leq (1+\delta)\frac{\left[\sum_{j=2}^\infty(1+(j-1)\lambda)^\beta|a_j|(j-1)\lambda(1-\mu)\right]}{1-\sum_{j=2}^\infty(1+(j-1)\lambda)^\beta|a_j|(1+\mu(j-1)\lambda)}$$

This last expression is bounded above by $1-\alpha$ if

$$\sum_{j=2}^\infty[1+(j-1)\lambda]^\beta\left[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}\right]a_j \leq 1-\alpha.$$

Remark 2.1. If $f \in TS_p^\lambda(\mu,\alpha,\beta,\delta)$, then

$$a_j \leq \frac{1-\alpha}{[1+(j-1)\lambda]^\beta\left[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}\right]}$$

for $j = 2, 3, 4, \ldots$ and equality holds for

$$f(z) = z - \frac{1-\alpha}{[1+(j-1)\lambda]^\beta\left[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}\right]}z^j.$$

When $\mu = 0$ and $\delta = 1$, we have the following corollary proved in [10].

Corollary 2.1. *The necessary and sufficient condition for a function $f(z)$ in the form (2) to be in the class $TS_p^\lambda(\alpha,\beta)$, with $\alpha \in [0,1)$, is:*

$$\sum_{j=2}^\infty[1+(j-1)\lambda]^\beta[2(j-1)\lambda+1-\alpha]a_j \leq 1-\alpha$$

3. Growth and Distortion Theorem

Theorem 3.1. *If $f(z) \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$, then*

$$r - \left[\frac{1-\alpha}{(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)(1+\lambda)^{\beta}}\right] r^2 \leq |f(z)|$$
$$\leq r + r^2 \left[\frac{1-\alpha}{(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)(1+\lambda)^{\beta}}\right]$$

Equality holds for

$$f(z) = z - \frac{1-\alpha}{(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)(1+\lambda)^{\beta}} z^2 \text{ at } z = \pm r.$$

Proof. By Theorem 2.1, $f(z) \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$ if and only if

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}[\delta\{(j-1)\lambda + \mu\lambda - \mu\lambda j\} - \{\alpha + \alpha\mu j - \alpha\mu - 1 - (j-1)\lambda\}]a_j \leq 1 - \alpha$$

or equivalently

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}\{j(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda) - [\delta\lambda - \delta\mu\lambda + \alpha - \alpha\mu + \lambda - 1]\}a_j \leq 1 - \alpha.$$

This implies $$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}a_j(j-t) \leq 1 - t, \tag{4}$$

where $t = 1 - \left(\frac{1-\alpha}{\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda}\right)$. But

$$(1+\lambda)^{\beta}(2-t)\sum_{j=2}^{\infty}a_j \leq \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}a_j(j-t) \leq 1 - t$$

This last inequality follows from (4). We obtain,

$$|f(z)| \leq r + \sum_{j=2}^{\infty}a_j r^j \leq r + r^2\sum_{j=2}^{\infty}a_j \leq r + r^2\left(\frac{1-t}{(2-t)(1+\lambda)^{\beta}}\right).$$

Similarly,

$$|f(z)| \geq r - \sum_{j=2}^{\infty}a_j r^j \leq r - r^2\sum_{j=2}^{\infty}a_j \leq r - r^2\left(\frac{1-t}{(2-t)(1+\lambda)^{\beta}}\right).$$

Thus,

$$r - \left(\frac{1-t}{(2-t)(1+\lambda)^{\beta}}\right) r^2 \leq |f(z)| \leq r + r^2 \left(\frac{1-t}{(2-t)(1+\lambda)^{\beta}}\right)$$

that is,

$$r - \left[\frac{1-\alpha}{(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)(1+\lambda)^{\beta}}\right] r^2 \leq |f(z)|$$
$$\leq r + r^2 \left[\frac{1-\alpha}{(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)(1+\lambda)^{\beta}}\right].$$

Hence the result.

Corollary 3.1. *When $\mu = 0$, $\delta = 1$ we obtain for the functions $f(z) \in TS_p^{\lambda}(\alpha, \beta)$, with $\alpha \in [0, 1)$:*

$$r - \left(\frac{1-\alpha}{(1+\lambda)^{\beta}[2\lambda + 1 - \alpha]}\right) r^2 \leq |f(z)| \leq r + r^2 \left(\frac{1-\alpha}{(1+\lambda)^{\beta}[2\lambda + 1 - \alpha]}\right)$$

, $|z| = r$.

This corollary is due to [10].

Theorem 3.2. *If $f \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$, then*

$$1 - \left[\frac{2(1-\alpha)}{(1+\lambda)^{\beta}(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)}\right] r \leq |f'(z)|$$
$$\leq 1 + r \left[\frac{2(1-\alpha)}{(1+\lambda)^{\beta}(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)}\right]$$

Proof. Since $f \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$, we have

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta} a_j (j - t) \leq 1 - t,$$

where $t = 1 - \left(\frac{1-\alpha}{\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda}\right)$.

In view of Theorem 3.1, we have

$$\sum_{j=2}^{\infty} j a_j = \sum_{j=2}^{\infty} (j-t) a_j + t \sum_{j=2}^{\infty} a_j$$
$$\leq \frac{1-t}{(1+\lambda)^{\beta}} + t \left(\frac{1-t}{(2-t)(1+\lambda)^{\beta}}\right)$$
$$\leq \frac{2(1-t)}{(1+\lambda)^{\beta}(2-t)}$$

$$|f'(z)| \leq 1 + \sum_{j=2}^{\infty} ja_j |z|^{j-1} \leq 1 + r \sum_{j=2}^{\infty} ja_j \leq 1 + r \left[\frac{2(1-t)}{(1+\lambda)^{\beta}(2-t)} \right]$$

Similarly,

$$|f'(z)| \geq 1 - \sum_{j=2}^{\infty} ja_j |z|^{j-1} \geq 1 - r \sum_{j=2}^{\infty} ja_j \geq 1 - r \left[\frac{2(1-t)}{(1+\lambda)^{\beta}(2-t)} \right]$$

Hence,

$$1 - r \left[\frac{2(1-t)}{(1+\lambda)^{\beta}(2-t)} \right] \leq |f'(z)| \leq 1 + r \left[\frac{2(1-t)}{(1+\lambda)^{\beta}(2-t)} \right]$$

Substituting for t, we have

$$1 - \left[\frac{2(1-\alpha)}{(1+\lambda)^{\beta}(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)} \right] r \leq |f'(z)|$$
$$\leq 1 + r \left[\frac{2(1-\alpha)}{(1+\lambda)^{\beta}(\delta\lambda - \delta\mu\lambda - \alpha\mu + \lambda - \alpha + 1)} \right].$$

Corollary 3.2. *When $\mu = 0$, $\delta = 1$ we obtain for the functions $f(z) \in TS_p^{\lambda}(\alpha, \beta)$, with $\alpha \in [0, 1)$:*

$$1 - \left[\frac{2(1-\alpha)}{(1+\lambda)^{\beta}(2\lambda - \alpha + 1)} \right] r \leq |f'(z)| \leq 1 + \left[\frac{2(1-\alpha)}{(1+\lambda)^{\beta}(2\lambda - \alpha + 1)} \right]$$

r , $|z| = r$.

This corollary is due to [10].

4. Closure Theorem

Theorem 4.1. *If $f_1(z) = z$ and*

$$f_j(z) = z - \frac{(1-\alpha)}{E(j, \lambda, \beta, \delta, \mu, \alpha)} z^j$$

where

$$E(j, \lambda, \beta, \delta, \mu, \alpha) = (1+(j-1)\lambda)^{\beta}[\delta\{(j-1)\lambda + \mu\lambda - \mu\lambda j\} - \{\alpha + \alpha\mu j - \alpha\mu - 1 - (j-1)\lambda\}],$$

then $f \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$ if and only if it can be expressed in the form $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z)$, where $\lambda_j \geq 0$ and $\sum_{j=1}^{\infty} \lambda_j = 1$.

Proof. Let $f(z) = \sum_{j=2}^{\infty} \lambda_j f_j(z)$, $\lambda_j \geq 0$, $j = 1, 2, \ldots$ with $\sum_{j=1}^{\infty} \lambda_j = 1$. We have

$$\begin{aligned} f(z) &= \sum_{j=1}^{\infty} \lambda_j f_j(z) \\ &= \lambda_1 z + \sum_{j=2}^{\infty} \lambda_j \left[z - \frac{(1-\alpha)}{E(j,\lambda,\beta,\delta,\mu,\alpha)} z^j \right] \\ &= \sum_{j=1}^{\infty} \lambda_j z - \sum_{j=2}^{\infty} \lambda_j \left[\frac{(1-\alpha)}{E(j,\lambda,\beta,\delta,\mu,\alpha)} z^j \right] \\ &= z - \sum_{j=2}^{\infty} \lambda_j \left[\frac{(1-\alpha)}{E(j,\lambda,\beta,\delta,\mu,\alpha)} z^j \right] \end{aligned}$$

We have,

$$\begin{aligned} &\sum_{j=2}^{\infty} \frac{E(j,\lambda,\beta,\delta,\mu,\alpha)\lambda_j(1-\alpha)}{E(j,\lambda,\beta,\delta,\mu,\alpha)} \\ &= (1-\alpha)\sum_{j=2}^{\infty} \lambda_j \\ &= (1-\alpha)(1-\lambda_1) \\ &< 1-\alpha. \end{aligned}$$

Therefore the condition (3) for $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z)$ is satisfied.
Thus $f(z) \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$.

Conversely, suppose $f(z) \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$,
$f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$ and take

$$\lambda_j = \frac{(1+(j-1)\lambda)^{\beta}[\delta\{(j-1)\lambda + \mu\lambda - \mu\lambda j\} - \{\alpha + \alpha\mu j - \alpha\mu - 1 - (j-1)\lambda\}]}{(1-\alpha)} a_j \geq 0,$$

$j = 2, 3, \ldots$, with $\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j$,
so that $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j$.
Using Theorem 2.1, we obtain

$$\begin{aligned} \sum_{j=2}^{\infty} \lambda_j &= \frac{1}{1-\alpha} \sum_{j=2}^{\infty} E(j,\lambda,\beta,\delta,\mu,\alpha) a_j \\ &< \frac{1}{1-\alpha}(1-\alpha) = 1. \end{aligned}$$

Hence $1-\lambda_1<1$ or $\lambda_1>0$.

Corollary 4.1. *The extreme points of* $TS_p^{\lambda}(\mu,\alpha,\beta,\delta)$ *are* $f_1(z)=z$ *and*

$$f_j(z)=z-\frac{(1-\alpha)}{E(j,\lambda,\beta,\delta,\mu,\alpha)}z^j$$

When $\mu=0$, $\delta=1$, *we obtain* $f_j(z)=z-\frac{(1-\alpha)}{(1+(j-1)\lambda)^{\beta}(2(j-1)\lambda+1-\alpha)}z^j$. *This result, regarding the subclass* $TS_p^{\lambda}(\alpha,\beta)$, *with* $\alpha\in[0,1)$, *is due to [10].*

5. Hadamard Product

Definition 5.1. **[7]** For two functions $f(z)$, $g(z)\in T$, $f(z)=z-\sum_{j=2}^{\infty}a_jz^j$ and $g(z)=z-\sum_{j=2}^{\infty}b_jz^j$, $(b_j\geq 0, j=2,3,\dots)$, the modified Hadamard product $f*g$ is defined by

$$(f*g)(z)=z-\sum_{j=2}^{\infty}a_jb_jz^j$$

To prove the next theorem, we need the following result due to Silverman [8] (for more details see [6]).

Theorem A. *If* $f(z)=z-\sum_{j=2}^{\infty}a_jz^j$, $a_j\geq 0$, $j=2,3,\dots$, $z\in U$, *then the next assertions are equivalent:*

(i) $\sum_{j=2}^{\infty}ja_j\leq 1$

(ii) $f\in T$

(iii) $f\in T^*$, *where* $T^*=T\cap S^*$ *and* S^* *is the well-known class of starlike functions.*

Theorem 5.1. *If* $f(z)=z-\sum_{j=2}^{\infty}a_jz^j\in TS_p^{\lambda}(\mu,\alpha,\beta,\delta)$ $(a_j\geq 0, j=2,3,\dots)$ *and* $g(z)\in T$, *with* $g(z)=z-\sum_{j=2}^{\infty}b_jz^j\in TS_p^{\lambda}(\mu,\alpha,\beta,\delta)$, $b_j\geq 0$, $j=2,3,\dots$, $\alpha\in[0,1)$, $\lambda\geq 0$, $\beta\geq 0$, *then* $f(z)*g(z)\in TS_p^{\lambda}(\mu,\alpha,\beta,\delta)$.

Proof. We have

$$\sum_{j=1}^{\infty}(1+(j-1)\lambda)^{\beta}[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}]a_j<1-\alpha$$

and

$$\sum_{j=1}^{\infty}(1+(j-1)\lambda)^{\beta}[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}]b_j<1-\alpha$$

We know that $f(z) * g(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j$.

From $g(z) \in T$, by using Theorem A, we have $\sum_{j=2}^{\infty} j b_j \leq 1$. We notice that $b_j < 1$, $j = 2, 3, \ldots$.

Thus

$$\sum_{j=1}^{\infty}(1+(j-1)\lambda)^{\beta}[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}]a_j b_j$$
$$<\sum_{j=1}^{\infty}(1+(j-1)\lambda)^{\beta}[\delta\{(j-1)\lambda+\mu\lambda-\mu\lambda j\}-\{\alpha+\alpha\mu j-\alpha\mu-1-(j-1)\lambda\}]a_j$$
$$<1-\alpha.$$

Hence $f(z) * g(z) \in TS_p^{\lambda}(\mu, \alpha, \beta, \delta)$, $\beta \geq 0$, $\lambda \geq 0$ and $\alpha \in [0, 1]$.

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