Acta Universitatis Apulensis No. 34/2013
ISSN: 1582-5329 pp. 17-33

NEW CONTROLLABILITY RESULTS FOR FRACTIONAL EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we use Caputo derivative to generate new sufficient conditions for the controllability of fractional evolution integro-differential equations in Banach spaces. These results are obtained using Banach contraction principle. Other results are also presented using Krasnoselskii theorem.

2000 *Mathematics Subject Classification*: 26A33, 93B05, 34K45.

1. Introduction

The fractional differential equations are emerged as a new branch of applied mathematics by which many physical phenomena in various fields of science and engineering can be modeled. Significant development in this area has been achieved for the last two decades. For details, we refer to [6,7,10,11,13]. Moreover, the study of impulsive fractional integro-differential equations is also of great importance. The study of such equations is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution [3,5,12,13]. Many researchers have discussed controllability of impulsive systems in Banach space. For more details, we refer the reader to [1,2,4,8]. Motivated by the works [14,16], the aim of this paper is to establish the controllability results for fractional evolution integro-differential systems in Banach spaces by using the fractional calculus and fixed point theorems. So, let us consider the fractional non linear integro-differential equations

$$\begin{cases} D^{\alpha}x(t) = A(t)x(t) + Bu(t) + f\left(t, x_t, \int_0^t a(t,s,x_s)\,ds\right), t \neq t_i, \\ t \in J, 0 < \alpha \leq 1, \\ x(s) + [g(x_{t_1}, ..., x_{t_p})](s) = \theta(s), s \in [-h, 0], \\ \Delta x|_{t=ti} = I_i(x(t_i^-)), i = 1, 2, ..., m, \end{cases} \quad (1)$$

where D^{α} is the Caputo derivative, the variable x takes values in a Banach space X and the control function u is given in $L^2(J,V)$; V as a Banach space and $J := [0,b]$, $\Omega := \{(t,s); 0 \le s \le t \le b\}$. We suppose that A is a closed linear densely defined operator in X, and B is a bounded linear operator from V into X.

Further, we suppose $f : J \times X \times X \to X$, $a : \Omega \times X \to X$, $I_i : X \to X, \Delta x|_{t=ti} = x(t_i^+) - x(t_i^-), i = 1, ..., m,\ 0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = b$, $g : [PC([-h,0],X)]^p \to X$ are given functions. The function $x_t : (-h,0] \to X$ is defined by $x_t(\eta) = x(t+\eta)$, for $t \in [0,b], \eta \in [-h,0]$.
We also take $J_0 := [0,t_1]$, $J_i = (t_i, t_{i+1}]$, $i = 1, 2, ..., m$, $J' := [0,b]\backslash\{t_1, ..., t_m\}$, $PC([-h,b],X) := \{x; x$ is a function from $[-h,b]$ into X such that $x(t)$ is continuous at $t \neq t_i$, left continuous at $t = t_i$ and the right limit $x(t_i^+)$ exists for $i = 1, 2, ..., m\}$, with norm

$$\| x \|_{PC} := \sup_{t \in J} \|x(t)\|.$$

It is to note that (1) is equivalent to

$$x(t) = \begin{cases} \theta(0) - [g(x_{t_1}, ..., x_{t_p})](0) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} A(\tau) x(\tau) d\tau \\ + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-\tau)^{\alpha-1} A(\tau) x(\tau) d\tau \\ + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} \left[Bu(\tau) + f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s) ds\right)\right] d\tau \\ + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-\tau)^{\alpha-1} \left[Bu(\tau) + f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s) ds\right)\right] d\tau \\ + \sum_{0<t_i<t} I_i\left(x\left(t_i^-\right)\right). \end{cases} \tag{2}$$

2. Preliminaries

In the following, we give the necessary notation and basic definitions which will be used in this paper.

Definition 1. *A real valued function* $f(t), t > 0$ *is said to be in the space* $C_\mu, \mu \in R$ *if there exists a real number* $p > \mu$ *such that* $f(t) = t^p f_1(t)$, *where* $f_1(t) \in C([0,\infty))$.

Definition 2. *A function* $f(t), t > 0$ *is said to be in the space* $C_\mu^n, n \in N$, *if* $f^{(n)} \in C_\mu$.

Definition 3. *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$, *for a function* $f \in C_\mu, (\mu \geq -1)$, *is defined as*

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau; \quad \alpha > 0, t > 0 \\ J^0 f(t) &= f(t). \end{aligned} \tag{3}$$

The fractional derivative of $f \in C_{-1}^n$ *in the Caputo's sense is defined as*

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, & n-1 < \alpha < n, n \in N^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{4}$$

.

For more details, we refer to [9, 15].
For the controllability, we give the following definition [1,2]:

Definition 4. *The system* (1) *is said to be controllable on the interval* J *if for every* $x_1 \in X$ *and* $[g(x_{t_1}, ..., x_{t_p}](s) \in PC([-h,b], X)$, *there exists a control* $u \in L^2(J,V)$ *such that the solution* $x(t)$ *of* (1) *satisfies* $x(0) = x_0$ *and* $x(b) = x_1$.

Let us now consider $B_r := \{x \in X, ||x|| \leq r, r > 0\}$. To study the controllability problem, we assume the following conditions:

(H_1) : $A(t)$ is a bounded linear operator on X, for each $t \in J$ and the function $t \to A(t)$ is continuous in the uniform operator topology and

$$\max_{t \in J} \|A(t)\| = M.$$

(H_2) : The linear operator $W : L^2(J;V) \to X$ defined by

$$Wu(t) = \frac{1}{\Gamma(\alpha)} \int_0^b (b-\tau)^{\alpha-1} Bu(\tau)\, d\tau$$

has an inverse operator W^{-1}, which takes values in $L^2(J;V)/kerW$ and there exists a positive constant $C > 0$ such that $\| BW^{-1} \| \leq C$, for every $x \in B_r$.

(H_3) : The function $f : J \times X \times X \to X$ is continuous and there exist constants $\beta > 0$,ß> 0, such that $||f(t, x_t, u_t) - f(t, y_t, v_t)|| \leq \beta (||x - y|| + \|u - v\|)\,; x, y, u, v \in X, t \in J$, ß $= \max_{t \in J} \|f(t, 0, 0)\|$.

(H_4) : For each $(t,s) \in \Omega$, the function $a : \Omega \times X \to X$ is continuous and there exist two constants $\omega > 0, \hat{\omega} > 0$, such that

$$\int_0^t \| a(t,s,x_s) - a(t,s,y_s) \| \, ds \leq \omega \| x - y \|, x, y \in X, t, s \in J$$

$$\hat{\omega} = \max \left\{ \int_0^t \| a(t,s,0) \| \, ds : t, s \in \Omega. \right\}$$

(H_5) : $I_i : X \to X$ is continuous and there exist constants ϖ_i, such that

$$\| I_i(x) - I_i(y) \| \leq \varpi_i \| x - y \|, i = 1, 2, ..., m, x, y \in X.$$

(H_6) : The function $g : [PC([-h,0],X)]^p \to X$ is continuous and there exist two constants ϱ, $\hat{\varrho} > 0$, such that

$$\| [g(x_{t_1}, ..., x_{t_p}](s) - [g(y_{t_1}, ..., y_{t_p}](s) \| \leq \varrho \| x - y \|_{PC}; x, y \in PC([-h,b],X), s \in [-h,0]$$

$$\hat{\varrho} = \max\{\| [g(x_{t_1}, ..., x_{t_p}](s) \| : x, y \in PC([-h,b],X), s \in [-h,0]\}.$$

(H_7): There exists a positive constant $r > 0$, such that

$$\|\theta(0)\| + \hat{\varrho} + (m+1)\,\lambda\left[(M+\beta+b\omega)\,r + (\kappa + ß + b\hat{\omega})\right] + m\,(\varpi r + \hat{\varpi}) \leq r,$$

$$\kappa = C\left[\|x_1\| + \|\theta\,(0)\| + \hat{\varrho} \right.$$
$$\left. + (m+1)\,\lambda\left[(M+\beta+b\omega)\,r + ß + b\hat{\omega}\right] + m\,(\varpi r + \hat{\varpi}\right),$$

$$\lambda = \frac{b^\alpha}{\Gamma\,(\alpha+1)}.$$

3. Main Results

Using the well-known Banach fixed point theorem, we give the following theorem.
Theorem 1. *If the hypotheses* $(H_j)_{j=\overline{1,7}}$ *and*

$$\Lambda := \left[\varrho + (m+1)\,\lambda\left[\gamma + (M+\beta+b\omega)\right] + m\varpi\right] < 1 \tag{5}$$

are satisfied, then the problem (1) *is controllable on* J.

Proof. Using (H_2), we define the control u as follows:

$$u\,(t) = \begin{cases} w^{-1}[x_1 - \theta\,(0) + [g(x_{t_1},...,x_{t_p})](0) - \frac{1}{\Gamma(\alpha)} \sum\limits_{0<t_i<b} \int\limits_{t_{i-1}}^{t_i} (t_i-\tau)^{\alpha-1} A\,(\tau)\,x\,(\tau)\,d\tau \\ -\frac{1}{\Gamma(\alpha)} \int\limits_{t_i}^{b} (b-\tau)^{\alpha-1} A\,(\tau)\,x\,(\tau)\,d\tau \\ -\frac{1}{\Gamma(\alpha)} \sum\limits_{0<t_i<b} \int\limits_{t_i}^{t_i} (t_i-\tau)^{\alpha-1} f\left(\tau, x_\tau, \int\limits_0^\tau a\,(\tau,s,x_s)\,ds\right) d\tau \\ -\frac{1}{\Gamma(\alpha)} \int\limits_{t_i}^{b} (b-\tau)^{\alpha-1} f\left(\tau, x_\tau, \int\limits_0^\tau a\,(\tau,s,x_s)\,ds\right) d\tau \\ + \sum\limits_{0<t_i<t} I_i\left(x\left(t_i^-\right)\right)]\,(t)\,. \end{cases} \tag{6}$$

Now, we shall prove that the operator $\Phi : PC([-h,b], B_r) \to PC([-h,b], B_r)$ defined by

$$\Phi x(t) = \begin{cases} \theta(0) - [g(x_{t_1},...,x_{t_p})](0) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-\tau)^{\alpha-1} A(\tau) x(\tau) d\tau \\ + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i-\tau)^{\alpha-1} A(\tau) x(\tau) d\tau \\ + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i-\mu)^{\alpha-1} \phi(\mu,x) d\mu + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-\mu)^{\alpha-1} \phi(\mu,x) d\mu \\ + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i-\tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau,s,x_s) ds\right) d\tau \\ + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-\tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau,s,x_s) ds\right) d\tau + \sum_{0<t_i<t} I_i\left(x\left(t_i^-\right)\right) \end{cases} \tag{7}$$

has a fixed point. So, let us take

$$\phi(\mu,x) = \begin{cases} Bw^{-1}[x_1 - \theta(0) + [g(x_{t_1},...,x_{t_p})](0) - \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<b} \int_{t_{i-1}}^{t_i} (t_i-\tau)^{\alpha-1} A(\tau) x(\tau) d\tau \\ - \frac{1}{\Gamma(\alpha)} \int_{t_i}^{b} (b-\tau)^{\alpha-1} A(\tau) x(\tau) d\tau - \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<b} \int_{t_i}^{t_i} (t_i-\tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau,s,x_s) ds\right) d\tau \\ - \frac{1}{\Gamma(\alpha)} \int_{t_i}^{b} (b-\tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau,s,x_s) ds\right) d\tau \\ + \sum_{0<t_i<t} I_i\left(x\left(t_i^-\right)\right))](t). \end{cases} \tag{8}$$

Then, we have

$$\begin{gathered}\|\phi(\mu,x)\| \leq C\left[\|x_1\|+\|\theta(0)\|+\hat{\varrho}+\right.\\ \left.+(m+1)\lambda\left[(M+\beta+b\omega)r+ß+b\hat{\omega}\right]+m(\varpi r+\hat{\varpi}\right)=\kappa\end{gathered} \tag{9}$$

and

$$\|\phi(\mu,x)-\phi(\mu,y)\| \leq C\left[(m+1)\lambda\left[(M+\beta+b\omega)\right]+m(\varpi)\right]\|x-y\| \leq \gamma\|x-y\|.$$

1* We have to prove that $\Phi B_r \subset B_r$. Let $x \in B_r$, then we can write:

$$\begin{gathered}\|\Phi x((t))\|\\ \leq \|\theta(0)\|+\left\|[g(x_{t_1},...,x_{t_p})](0)\right\|+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)\|d\tau\\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)\|d\tau+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}\|\phi(\mu,x)\|d\mu\\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\mu)^{\alpha-1}\|\phi(\mu,x)\|d\mu+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)ds\right)\right\|d\tau\\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)ds\right)\right\|d\tau+\sum_{0<t_i<t}\left\|I_i\left(x\left(t_i^-\right)\right)\right\|.\end{gathered}$$

Consequently,

$$\begin{gathered}\|\Phi x((t))\|\\ \leq \|\theta(0)\|+\left\|[g(x_{t_1},...,x_{t_p})](0)\right\|+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)\|d\tau\\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)\|d\tau+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}\|\phi(\mu,x)\|d\mu\\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\mu)^{\alpha-1}\|\phi(\mu,x)\|d\mu\\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)ds\right)-f(\tau,0,0)+f(\tau,0,0)\right\|d\tau\\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)ds\right)-f(\tau,0,0)+f(\tau,0,0)\right\|d\tau\\ +\sum_{0<t_i<t}\left\|I_i\left(x\left(t_i^-\right)\right)-I_i(0)+I_i(0)\right\|.\end{gathered}$$

Therefore,

$$\|\Phi x\left((t)\right)\|$$

$$\leq \|\theta(0)\| + \left\|[g(x_{t_1},...,x_{t_p})](0)\right\| + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)\|d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)\|d\tau + \frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}\|\phi(\mu,x)\|d\mu$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\mu)^{\alpha-1}\|\phi(\mu,x)\|d\mu$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)\,ds\right)-f(\tau,0,0)\right\|d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\|f(\tau,0,0)\|d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)\,ds\right)-f(\tau,0,0)\right\|d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f(\tau,0,0)\,d\tau + \sum_{0<t_i<t}\left[\left\|I_i\left(x\left(t_i^-\right)\right)-I_i(0)\right\| + \|I_i(0)\|\right].$$

This implies that,

$$\|\Phi x\left((t)\right)\| \leq \|\theta(0)\| + \hat{\varrho} + (m+1)\,\lambda\left[(M+\beta+b\omega)\,r + (\kappa+ß+b\hat{\omega})\right] + m\left(\varpi r+\hat{\varpi}\right) \tag{10}$$

and then,

$$\|\Phi x\left((t)\right)\| \leq r. \tag{11}$$

Hence $\Phi B_r \subset B_r$, which means that the operator Φ maps B_r into itself.

2* Now we prove that Φ is a contraction mapping on B_r. Let $x, y \in B_r$, then we can write:

$$\|\Phi x(t) - \Phi y(t)\|$$

$$\leq \left\|[g(x_{t_1},...,x_{t_p}) - g(y_{t_1},...,y_{t_p})](0)\right\| + +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)-y(\tau)\|d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\|A(\tau)\|\|x(\tau)-y(\tau)\|d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}\|\phi(\mu,x)-\phi(\mu,y)\|d\mu$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\mu)^{\alpha-1}\|\phi(\mu,x)-\phi(\mu,y)\|d\mu$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)\,ds\right)-f\left(\tau,y_\tau,\int_0^\tau a(\tau,s,y_s)\,ds\right)\right\|d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)\,ds\right)-f\left(\tau,y_\tau,\int_0^\tau a(\tau,s,y_s)\,ds\right)\right\|d\tau$$

$$+\sum_{0<t_i<t}\left\|I_i\left(x\left(t_i^-\right)\right)-I_i\left(y\left(t_i^-\right)\right)\right\|.$$

Consequently,

$$\|\Phi x(t) - \Phi y(t)\| \leq [\varrho + (m+1)\lambda[\gamma + (M + \beta + b\omega)] + m\varpi] \|x - y\|. \quad (12)$$

Then,

$$\|\Phi x(t) - \Phi y(t)\| \leq \Lambda \|x - y\|. \quad (13)$$

Hence, the operator Φ has a unique fixed point which is a solution of (1). Consequently, the system (1) is controllable on J. Theorem 1 is thus proved.

Our second result is based on Krasnoselskii's fixed point theorem [11].
Let us consider the following conditions:

(H_3') : i) For each $t \in J$, the function $f(t,\cdot,\cdot) : X \times X \to X$ is continuous.
ii) There exists a function $\beta_f(\cdot) \in L^1(J, R^+)$ such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq \beta_f(t)(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

$$ß = \max_{t\in J} \|f(t,0,0)\|$$

for any $t \in J$, x_l, $y_i \in X$ for $l = 1, 2$.
iii) The function $f : J \times X \times X \to X$ is compact.

(H_4') : i) For each $t, s \in J$, the function $a(t,s,\cdot) : X \to X$ is continuous .
ii) There exists an integrable function $\omega_a : J \times J \to [0,\infty)$ such that

$$\|a(t,s,x) - a(t,s,y)\| \leq \omega_a(t,s)\|x - y\|, t,s \in J, x,y \in X,$$

$$\hat{\omega} = \max\left\{\int_0^t \| a(t,s,0) \| ds : t,s \in \Omega.\right\}.$$

(H_5'): i) $I_i : X \to X$ is continuous and there exist constants ϖ_i such that

$$\| I_i(x) - I_i(y) \| \leq \varpi_i \| x - y \|, i = 1,2,...,m, x,y \in X.$$

ii) $I_i : X \to X$ is compact.

(H_7'): There exists a positive constant $r > 0$, such that

$$\|\theta(0)\| + \hat{\varrho} + (m+1)\,\lambda\left[\left(M + \|\beta_f\|_{L^1} + b\,\|\omega_a\|_{L^1}\right)\right.$$
$$\left. r + \left(\kappa + \|\beta_f\|_{L^1} + b\,\|\omega_a\|_{L^1}\right) + m\,(\varpi r + \hat{\varpi}) \le r,\right.$$

$\kappa =$

$$C\left[\|x_1\| + \|\theta\,(0)\| + \hat{\varrho} + (m+1)\,\lambda\left[\left(M + \|\beta_f\|_{L^1} + b\,\|\omega_a\|_{L^1}\right) r + ß + b\hat{\omega}\right] + m\,(\varpi r + \hat{\varpi})\right],$$

and

$$\lambda = \frac{b^{\alpha}}{\Gamma\,(\alpha+1)}.$$

(H_8): There exists a constant Υ such that

$$\Upsilon = \varrho + (m+1)\,\lambda\left[M + C\left[(m+1)\,\lambda\left[\left(M + \|\beta_f\|_{L^1} + b\,\|\omega_a\|_{L^1}\right)\right] + m\,(\varpi)\right]\right].$$

Theorem 2. *(Krasnoselskii Theorem) Let K be a closed convex and nonempty subset of a Banach space X. Let P, Q be the operators such that*
(i) $Px + Qy \in K, x, y \in K$.
(ii) P is compact and continuous.
(iii) Q is a contraction mapping.
Then there exists x^*, such that

$$x^* = Px^* + Qx^*.$$

Theorem 3. *If the hypotheses (H_1), (H_6), (H_8), $(H_j^{'})_{j=\overline{3,5}}$ and $(H_7^{'})$ are satisfied and if $\Upsilon < 1$, then the system (1) is controllable on J.*

Proof. On B_r, we define the operators R and S as:

$$\begin{cases} Rx\left(t\right)=\theta(0)-[g(x_{t_1},...,x_{t_p})](0)+\frac{1}{\Gamma(\alpha)}\int\limits_{t_i}^{t}\left(t-\tau\right)^{\alpha-1}A\left(\tau\right)x\left(\tau\right)d\tau \\ \qquad +\frac{1}{\Gamma(\alpha)}\sum\limits_{0<t_i<t}\int\limits_{t_{i-1}}^{t_i}\left(t_i-\tau\right)^{\alpha-1}A\left(\tau\right)x\left(\tau\right)d\tau+ \\ \qquad \frac{1}{\Gamma(\alpha)}\sum\limits_{0<t_i<t}\int\limits_{t_{i-1}}^{t_i}\left(t_i-\mu\right)^{\alpha-1}\phi\left(\mu,x\right)d\mu \\ \qquad +\frac{1}{\Gamma(\alpha)}\int\limits_{t_i}^{t}\left(t-\mu\right)^{\alpha-1}\phi\left(\mu,x\right)d\mu \end{cases} \tag{14}$$

and

$$\begin{cases} Sx\left(t\right)=\frac{1}{\Gamma(\alpha)}\sum\limits_{0<t_i<t}\int\limits_{t_{i-1}}^{t_i}\left(t_i-\tau\right)^{\alpha-1}f\left(\tau,x_\tau,\int\limits_0^\tau a\left(\tau,s,x_s\right)ds\right)d\tau \\ +\frac{1}{\Gamma(\alpha)}\int\limits_{t_i}^{t}\left(t-\tau\right)^{\alpha-1}f\left(\tau,x_\tau,\int\limits_0^\tau a\left(\tau,s,x_s\right)ds\right)d\tau+\sum\limits_{0<t_i<t}I_i\left(x\left(t_i^-\right)\right). \end{cases} \tag{15}$$

For x, $y\in B_r$, we have

$$\left\|Rx\left(t\right)+Sy\left(t\right)\right\|\leq\left\|Rx\left(t\right)\right\|+\left\|Sx\left(t\right)\right\|. \tag{16}$$

Then we can write

$$\|Rx(t)+Sy(t)\| \leq \|\theta(0)\| + \left\|[g(x_{t_1},...,x_{t_p})](0)\right\| + \left\|\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}A(\tau)x(\tau)\,d\tau\right\|$$

$$+\left\|\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}A(\tau)x(\tau)\,d\tau\right\|$$

$$+\left\|\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}\phi(\mu,x)\,d\mu\right\|$$

$$+\left\|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_i}^{t}(t-\mu)^{\alpha-1}\phi(\mu,x)\,d\mu\right\|$$

$$+\left\|\tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)\,ds\right)d\tau\right\|$$

$$+\left\|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_i}^{t}(t-\tau)^{\alpha-1}f\left(\tau,x_\tau,\int_0^\tau a(\tau,s,x_s)\,ds\right)d\tau\right\| + \left\|\textstyle\sum_{0<t_i<t}I_i\left(x\left(t_i^-\right)\right)\right\|.$$

It follows then that

$$\|Rx(t)+Sy(t)\| \leq \|\theta(0)\| + \hat{\varrho} + (m+1)\lambda\left(M+\|\beta_f\|_{L^1}+b\|\omega_a\|_{L^1}\right)r$$

$$+(m+1)\lambda\left(\kappa+\|\beta_f\|_{L^1}+b\|\omega_a\|_{L^1}\right)+m(\varpi r+\hat{\varpi}).$$

Consequently,

$$\|Rx(t)+Sy(t)\| \leq r.$$

Hence, $Rx+Sy\in B_r$.
On other hand, we have

$$\|Rx(t)-Ry(t)\| = \| \, [\theta(0)-[g(x_{t_1},...,x_{t_p})](0)+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}A(\tau)x(\tau)\,d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}A(\tau)x(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}\phi(\mu,x)\,d\mu$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\mu)^{\alpha-1}\phi(\mu,x)\,d\mu]-[\theta(0)-[g(y_{t_1},...,y_{t_p})](0)+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}A(\tau)y(\tau)\,d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}A(\tau)y(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}\phi(\mu,y)\,d\mu$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\mu)^{\alpha-1}\phi(\mu,x)\,d\mu]\|. \tag{17}$$

Therefore,

$$\|Rx(t)-Ry(t)\| \leq \left\|[g(y_{t_1},...,y_{t_p})](0)-[g(x_{t_1},...,x_{t_p})](0)\right\|$$
$$+\left\|\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}A(\tau)(x(\tau)-y(\tau))\,d\tau\right\|$$
$$+\left\|\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}A(\tau)(x(\tau)-y(\tau))\,d\tau\right\|$$
$$+\left\|\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\mu)^{\alpha-1}[\phi(\mu,x)-\phi(\mu,y)]\,d\mu\right\|$$
$$+\left\|\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\mu)^{\alpha-1}[\phi(\mu,x)-\phi(\mu,y)]\,d\mu\right\|$$
$$\leq \left[\varrho+(m+1)\lambda\left[M+C\left[(m+1)\lambda\left[\left(M+\|\beta_f\|_{L^1}+b\left\|\omega_a\right\|_{L^1}\right)\right]+m(\varpi)\right]\right]\right]\|x-y\|$$
$$\leq \Upsilon\,\|x-y\|\,.$$

Since $\Upsilon<1$, then the operator R is a contraction.
Now, we shall prove that the operator S is completely continuous on B_r: First, we show that S is continuous on B_r. So, let $\{x^n(t)\}_0^\infty\subset B_r$, with $x^n\to x$ in B_r. Then, there exists $r>0$, such that $\|x_n(t)\|\leq r$, for all n. Hence $x_n\in B_r$ and $x\in B_r$. The hypotheses $(H_3^{'}),(H_4^{'})$ and $(H_5^{'})$ imply that:

i) I_i is continuous for each $i = 1, 2, ..., m$.
ii) $f\left(t, x_t^n, \int_0^t a\left(t, s, x_s^n\right) ds\right) \to f\left(t, x_t, \int_0^t a\left(t, s, x_s\right) ds\right)$ for each $t \in J$ and

$$\left\| f\left(t, x_t^n, \int_0^t a\left(t, s, x_s^n\right) ds\right) \to f\left(t, x_t, \int_0^t a\left(t, s, x_s\right) ds\right) \right\| \leq 2r \left\|\beta_f\right\|_{L^1}.$$

Using Lebesgue dominated convergence theorem, yields

$$\left\|Sx^n(t) - Sx(t)\right\| =$$

$$\left\| \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} \left(f\left(\tau, x_\tau^n, \int_0^\tau a\left(\tau, s, x_s^n\right) ds\right) - f\left(\tau, x_\tau, \int_0^\tau a\left(\tau, s, x_s\right) ds\right)\right) d\tau \right.$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t - \tau)^{\alpha-1} \left(f\left(\tau, x_\tau^n, \int_0^\tau a\left(\tau, s, x_s^n\right) ds\right) - f\left(\tau, x_\tau, \int_0^\tau a\left(\tau, s, x_s\right) ds\right)\right) d\tau$$

$$\left. + \sum_{0<t_i<t} \left(I_i\left(x^n\left(t_i^-\right)\right) - I_i\left(x\left(t_i^-\right)\right)\right) \right\|.$$

Furthermore,

$$\left\|Sx^n(t) - Sx(t)\right\| \leq P + Q + Z,$$

where,

$$P = \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} \left\| f\left(\tau, x_\tau^n, \int_0^\tau a\left(\tau, s, x_s^n\right) ds\right) - f\left(\tau, x_\tau, \int_0^\tau a\left(\tau, s, x_s\right) ds\right) \right\| d\tau,$$

$$Q = +\frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t - \tau)^{\alpha-1} \left\| f\left(\tau, x_\tau^n, \int_0^\tau a\left(\tau, s, x_s^n\right) ds\right) - f\left(\tau, x_\tau, \int_0^\tau a\left(\tau, s, x_s\right) ds\right) \right\| d\tau$$

and

$$Z = \sum_{0<t_i<t} \left\|I_i\left(x^n\left(t_i^-\right)\right) - I_i\left(x\left(t_i^-\right)\right)\right\|.$$

Consequently,

$$\left\|Sx^n(t) - Sx(t)\right\| \to 0, n \to \infty.$$

Hence, S is continuous on B_r.

Now, we prove that S is relatively compact as well as equi-continuous on X. To prove the compactness of S, we shall prove that $S(B_r) \subseteq PC([-h, b], X)$ is equi-continuous and $S(B_r)(t)$ is pre-compact for any $r > 0$, $t \in J$. Let $x \in B_r$ and $t + k \in J$. Then we can write

$$\|Sx(t+k) - Sx(t)\| \leq \| \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t+k} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s)\, ds\right) d\tau$$

$$+\tfrac{1}{\Gamma(\alpha)} \int_{t_i}^{t+k} ((t+k) - \tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s)\, ds\right) d\tau$$

$$+\textstyle\sum_{0<t_i<t+k} I_i\left(x\left(t_i^-\right)\right)$$

$$-\tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s)\, ds\right) d\tau$$

$$+\tfrac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t - \tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s)\, ds\right) d\tau + \textstyle\sum_{0<t_i<t} I_i\left(x\left(t_i^-\right)\right) \|.$$

Therefore,

$$\|Sx(t+k) - Sx(t)\| \leq \| \frac{1}{\Gamma(\alpha)} \sum_{t<t_i<t+k} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s)\, ds\right) d\tau$$

$$+\tfrac{1}{\Gamma(\alpha)} [\int_{t_i}^{t+k} ((t+k) - \tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s)\, ds\right)$$

$$-\int_{t_i}^{t} (t - \tau)^{\alpha-1} f\left(\tau, x_\tau, \int_0^\tau a(\tau, s, x_s)\, ds\right) d\tau]$$

$$+\textstyle\sum_{t<t_i<t+h} I_i\left(x\left(t_i^-\right)\right) \|.$$

The inequality (27) is independent of x. Thus S is equi-continuous, and as $k \to 0$, the right hand side of this inequality tends to zero. Then $S(B_r)$ is relatively compact, and S is compact. Finally by Krasnoselskii theorem, there exists a fixed point x in B_r, such that $(\Phi x)(t) = x(t)$; this point is a solution of (1). It is clear that $(\Phi x)(b) = x(b) = x_1$, which implies that the system (1) is controllable on J.

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