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UNIVALENCE OF TWO INTEGRAL OPERATORS

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Abstract.In this paper we consider two integral operators and we study their univalence. The number of functions involved in the definition of the operators is taken as the entire part of the modulus of a complex number. Some particullar cases for construction of such integral operators are outlined and also some corollaries of univalence are derived.

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1. Introduction

Let be $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk and $\mathcal{A}$, the class of analytic functions normalized by the conditions $f(0) = f'(0) - 1 = 0$, defined as $f(z) = z + a_2 z^2 + ...$ Also let be $\mathcal{S} \subset \mathcal{A}$, the class of analytic functions which are univalent functions in the open unit disk.

We consider the following two general integral operators

$$F_{[|\delta|]}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\delta|]}(t)}{t}\right)^{\alpha_{[|\delta|]}} dt,$$

where $\delta \in \mathbb{C}$, $|\delta| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, $f_i \in \mathcal{A}$, $i = \overline{1,[|\delta|]}$ and

$$G_{[|\gamma|]}(z) = \left[\gamma \int_0^z t^{\gamma-1} \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt\right]^{\frac{1}{\gamma}},$$

$\gamma \in \mathbb{C}$, $|\gamma| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, $f_i \in \mathcal{A}$, $i = \overline{1,[|\gamma|]}$.

Remarks 1.1. If $|\delta| \in [0,1)$ we have $[|\delta|] = 0$ hence δ can not be used to count the number of functions involved in the definition of the operators.

Remarks 1.2. The parameter δ can be in particular chosen in conection with the other parameters from the definition of the integral operators, $\alpha_1, \alpha_2, ..., \alpha_{[|\delta|]}$. For example we can take $\mathrm{Re}\alpha_1 \cdot ... \cdot \mathrm{Re}\alpha_{[|\delta|]} = \mathrm{Re}\delta$ or $\mathrm{Re}\alpha_1 + ... + \mathrm{Re}\alpha_{[|\delta|]} = \mathrm{Re}\delta$ or $\alpha_1 + ... + \alpha_{[|\delta|]} = \gamma$. For such particullar cases further discutions should be done with respect to the existence of the integral operator based on those parameters.

In what follows we study the univalence of the integral operators $F_{[|\delta|]}$ and $G_{[|\gamma|]}$. In order to obtain univalence conditions for the above mentioned integral operators, we need the following lemas:

Lemma 1.3.[2]. *Let α be a complex number, $\mathrm{Re}\alpha > 0$ and $f(z) = z + a_2z^2 + ...$ be a regular function in $\mathcal{U}$. If*

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, then for any complex number β, $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta\int_0^z t^{\beta-1}f'(t)dt\right]^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Lemma 1.4. [1]. *If the function g is regular in $\mathcal{U}$ and $|g(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in \mathcal{U}$, the following inequalities hold*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1-\overline{z}\xi}\right| \tag{1.1}$$

and

$$|g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2},$$

the equalities hold in the case $g(z) = \epsilon\frac{z+u}{1+\overline{u}z}$, where $|\epsilon| = 1$ and $|u| < 1$.

Remark 1.5. [1]. For $z = 0$, from inequality (1.1) we obtain for every $\xi \in \mathcal{U}$,

$$\left|\frac{g(\xi)-g(0)}{1-\overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and hence,

$$|g(\xi)| \leq \frac{|\xi|+|g(0)|}{1+|g(0)|\,|\xi|}.$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \le \frac{|z| + |a|}{1 + |a|\,|z|},$$

for all $z \in \mathcal{U}$.

2. Main results

Theorem 2.1. *Let $\delta \in \mathbb{C}$, $0 < \mathrm{Re}\delta \le 1$, $|\delta| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, for $i = \overline{1, [|\delta|]}$. If $f_i \in \mathcal{A}$, $f_i(z) = z + a_2^i z^2 + \dots$, for $i = \overline{1, [|\delta|]}$ and*

$$\left| \frac{f_i'(z)}{f_i(z)} - \frac{1}{z} \right| \le \mathrm{Re}\delta, \tag{2.1}$$

for all $i = \overline{1, [|\delta|]}$ and $z \in \mathcal{U}$,

$$|\alpha_1| + \dots + \left|\alpha_{[|\delta|]}\right| \le \left|\alpha_1 \cdot \dots \cdot \alpha_{[|\delta|]}\right| \le \frac{1}{\max\limits_{|z|\le 1}\left[\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]}, \tag{2.2}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + \dots + \alpha_{[|\delta|]} a_2^{[|\delta|]}\right|}{\left|\alpha_1 \cdot \dots \cdot \alpha_{[|\delta|]}\right|},$$

then $F_{[|\delta|]} \in \mathcal{S}$.

Proof. Let consider the function

$$h(z) = \frac{1}{\left|\alpha_1 \cdot \dots \cdot \alpha_{[|\delta|]}\right|} \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)} = \frac{1}{\left|\alpha_1 \cdot \dots \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i \left(\frac{f_i'(z)}{f_i(z)} - \frac{1}{z} \right), \tag{2.3}$$

for all $z \in \mathcal{U}$ and we have

$$h(0) = \frac{1}{\left|\alpha_1 \cdot \dots \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i a_2^i.$$

By using the relation (2.1) and the first part from the inequality (2.2) we obtain that $|h(z)| < 1$ and $|h(0)| = |c|$.

Further if we apply Remark 1.5 for the function h we get

$$\left|\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\cdot z\cdot\frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right|\leq\left|\alpha_1\cdot\ldots\cdot\alpha_{[|\delta|]}\right|\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\cdot|z|\cdot\frac{|z|+|c|}{1+|c|\,|z|},\qquad(2.4)$$

for all $z\in\mathcal{U}$.

Now we consider the function $H:[0,\ 1]\to\mathbb{R}$ defined by

$$H(x)=\frac{1-x^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}x\frac{x+|c|}{1+|c|\,x};\ x=|z|\,.\qquad(2.5)$$

and we have $\max\limits_{x\in[0,1]}H(x)>0$.

Using this result togheter with the previous inequality and with the hypothesis conditions we get

$$\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\cdot\left|z\cdot\frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right|\leq 1,$$

for all $z\in\mathcal{U}$, so $F_{[|\delta|]}\in\mathcal{S}$.

If we take different values of δ we can obtain interesting univalence properties for particullar integral operators.

Let $\delta=\frac{1}{2}+i$. We obtain the next particullar case:

Corollary 2.2.*Let $\alpha\in\mathbb{C}$. If $f(z)=z+a_2z^2+...\in\mathcal{A}$ and*

$$\left|\frac{f'(z)}{f(z)}-\frac{1}{z}\right|\leq\frac{1}{2}$$

for all $z\in\mathcal{U}$

$$|\alpha|\leq\frac{1}{2\max\limits_{|z|\leq 1}\left[(1-|z|)\,|z|\cdot\frac{|z|+|a_2|}{1+|a_2||z|}\right]},$$

then

$$F_1(z)=\int\limits_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt$$

is univalent.

Let $\delta=\frac{1}{2}+2i$. We obtain the next particullar case:

Corollary 2.3. *Let $\alpha_1, \alpha_2 \in \mathbb{C}$. If $f_1(z) = z + a_2^1 z^2 + ... \in \mathcal{A}$, $f_2(z) = z + a_2^2 z^2 + ... \in \mathcal{A}$ and*

$$\left|\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right| \leq \frac{1}{2}, \ \left|\frac{f_2'(z)}{f_2(z)} - \frac{1}{z}\right| \leq \frac{1}{2},$$

for all $z \in \mathcal{U}$

$$\frac{1}{|\alpha_1|} + \frac{1}{|\alpha_2|} \leq 1,$$

$$|\alpha_1 \alpha_2| \leq \frac{1}{2 \max\limits_{|z| \leq 1} \left[(1 - |z|)\,|z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]},$$

where $|c| = \frac{|\alpha_1 a_2^1 + \alpha_2 a_2^2|}{|\alpha_1 \alpha_2|}$ then

$$F_2(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \left(\frac{f_2(t)}{t}\right)^{\alpha_2} dt$$

is univalent.

Theorem 2.4. *Let γ, $\delta \in \mathbb{C}$, $|\gamma| \notin [0, 1)$, $\alpha_i \in \mathbb{C}$, for $i = \overline{1, [|\gamma|]}$. If $f_i \in \mathcal{A}$, $f_i(z) = z + a_2^i z^2 + ...$, for $i = \overline{1, [|\gamma|]}$ and*

$$\left|\frac{f_i'(z)}{f_i(z)} - \frac{1}{z}\right| \leq 1, \tag{2.6}$$

for all $i = \overline{1, [|\delta|]}$ and $z \in \mathcal{U}$,

$$\mathrm{Re}\gamma \geq \mathrm{Re}\delta > 0,$$

$$|\alpha_1| + ... + \left|\alpha_{[|\delta|]}\right| \leq \left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]}, \tag{2.7}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\gamma|]} a_2^{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|},$$

then the integral operator $G_{[|\gamma|]}$ is in the class $\mathcal{S}$.

Proof. Let be the function

$$h(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt. \tag{2.8}$$

We denote by p the following function

$$p(z)=\frac{1}{\left|\alpha_1\cdot...\cdot\alpha_{[|\gamma|]}\right|}\cdot\frac{h''(z)}{h'(z)}=\frac{1}{\left|\alpha_1\cdot...\cdot\alpha_{[|\gamma|]}\right|}\sum_{i=1}^{[|\gamma|]}\alpha_i\left(\frac{f_i'(z)}{f_i(z)}-\frac{1}{z}\right).$$

for all $z\in\mathcal{U}$.

From (2.6) and (2.7) we get that $|p(z)|<1$ and

$$|p(0)|=\frac{\left|\alpha_1 a_2^1+...+\alpha_{[|\gamma|]}a_2^{[|\gamma|]}\right|}{\left|\alpha_1\cdot...\cdot\alpha_{[|\gamma|]}\right|}=|c|\,.$$

Here, applying Remark 1.5 for the function h we obtain that

$$\left|\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\cdot z\cdot\frac{h''(z)}{h'(z)}\right|\le\left|\alpha_1\cdot...\cdot\alpha_{[|\gamma|]}\right|\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\cdot|z|\cdot\frac{|z|+|c|}{1+|c|\,|z|}\tag{2.9}$$

for all $z\in\mathcal{U}$.

Considering the same function H, with the similarly properties and following the same steps as in the previous theorem, we obtain that $G_{[|\gamma|]}\in\mathcal{S}$.

Let $\gamma=1+i$. We obtain the next particullar case:

Corollary 2.5.*Let $\alpha,\delta\in\mathbb{C}$. If $f(z)=z+a_2z^2+...\in\mathcal{A}$ and*

$$\left|\frac{f'(z)}{f(z)}-\frac{1}{z}\right|\le 1$$

for all $z\in\mathcal{U}$, $0<\mathrm{Re}\delta\le 1$,

$$|\alpha|\le\frac{1}{\max\limits_{|z|\le 1}\left[\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\cdot|z|\cdot\frac{|z|+|a_2|}{1+|a_2||z|}\right]},$$

then

$$G_1(z)=\left[(1+i)\int_0^z t^i\left(\frac{f(t)}{t}\right)^{\alpha}dt\right]^{\frac{1}{1+i}}$$

is univalent.

Let $\gamma=2+i$. We obtain the next particullar case:

Corollary 2.6. *Let* $\alpha_1, \alpha_2, \delta \in \mathbb{C}$. *If* $f_1(z) = z + a_2^1 z^2 + ... \in \mathcal{A}$, $f_2(z) = z + a_2^2 z^2 + ... \in \mathcal{A}$ *and*

$$\left| \frac{f_1'(z)}{f_1(z)} - \frac{1}{z} \right| \leq 1, \ \left| \frac{f_2'(z)}{f_2(z)} - \frac{1}{z} \right| \leq 1,$$

for all $z \in \mathcal{U}$, $0 < \mathrm{Re}\delta \leq 2$

$$\frac{1}{|\alpha_1|} + \frac{1}{|\alpha_2|} \leq 1,$$

$$|\alpha_1 \alpha_2| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c||z|} \right]},$$

where $|c| = \frac{|\alpha_1 a_2^1 + \alpha_2 a_2^2|}{|\alpha_1 \alpha_2|}$ *then*

$$G_2(z) = \left[(2+i) \int_0^z t^{1+i} \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \left(\frac{f_2(t)}{t} \right)^{\alpha_2} dt \right]^{\frac{1}{2+i}}$$

is univalent.

Remark 2.7. We observe that if we take different complex values for γ, having the same entire part of modulus, we obtain the integral operator with the same numbers of functions, univalent but with some different univalence conditions.

References

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