Acta Universitatis Apulensis No. 33/2013
ISSN: 1582-5329 pp. 165-175

SOLVING BRACHISTOCHRONE PROBLEM USING HOMOTOPY ANALYSIS METHOD

Mehrdad Lakestani and Mahnaz Kazemian

Abstract. In this work, analytical technique, is applied to obtain an approximate analytical solution of the brachistochrone problem. The main objective is to find the solution of an brachistochrone problem. This work is done using homotopy analysis method. The method is general, easy to implement, and yields very accurate results with few computations. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series.

2000 *Mathematics Subject Classification*: 49K15, 49M30.

1. Introduction

In the large number of problems arising in analysis, mechanics, geometry, and so forth, it is necessary to determine the maximal and minimal of a certain functional. Because of the important role of this subject in science and engineering, considerable attention has been received on this kind of problems. such problems are called variational problems.
The problem of brachistochrone is proposed in 1696 by Johann Bernoulli which is required to find the line connecting two certain points A and B that do not lie on a vectorial line and possessing the property that a moving particle slides down this line from A to B in the shortest time. This problem was solved by Johann Bernoulli, Jacob Bernoulli, Leibnits, and Newton. It is shown that the solution of this problem is a cycloid. The solution of the brachistochrone problem is often cited as the origin of the calculus of variations as suggested in [23].

The classical brachistochrone problem deals with a mass moving along a smooth path in a uniform gravitational field. A mechanical analogy is the motion of a bead sliding down a frictionless wire. The solution to thus problem was obtained by various methods such as the gradient method [2], successive sweep algorithm in [1,3] the classical Chebyshev method [24], multistage Monte Carlo method [21] and legendre wavelet method [19].

More historical commens about variational problems are found in [7,8].
The simplest form of a variational problem can be considered as

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx, \tag{1}$$

where v is the funvtional that its extremum must be found. To find the extreme value of v, the boundary points of the admissible curves are known in the following form:

$$y(x_0) = \alpha, \qquad y(x_1) = \beta. \tag{2}$$

One of the popular methods for solving variational problems are direct methods. In these methods the variational problem is regarded as a limiting case of a finite number of variables. The direct method of Ritz and Galerkin has been investigated for solving variational problems in [7,8]. Chen and Hsiao [5] introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. Some orthogonal polynomials are applied on variational problems to find continuous solutions for these problems [4, 10, 11]. Also Fourier series and Taylor series are applied to variational problems,respectively in [17], to find a continuous solution for this kind of problems. Other authors introduced the Legendre wavelets method [18], ratinalized Haar method [16], Adomian decomposition method [5], He s variational iteration method [22] and Chebyshev finite difference method for solving variational problems [20]. More historical comments about variational problems are found in [7,8].

In this paper, we consider homotopy analysis method for finding approximate solution of brachistochrone problem which is a famous problem in calculus of variations.
In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method(HAM). This method has been successfully applied to solve many types of nonlinear problems by others [13, 14, 15].

In this paper, the basic idea of the HAM is introduced and then is applied to solve the brachistochrone problem. Also, the comparison is made with the exact solution. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series.

The outline of this paper is as follows: In section 2, we introduce the homotopy analysis method. In section 3, we introduce the brachistochrone problem. In section 4, the propose method is used brachistochrone problem to approximate the solution of the problem. As a result, the solution of the considered problem is introduced. Then we report our computational results and demonstrate the accuracy of the proposed numerical scheme by comparing our results with results obtained using other methods.

2.Homotopy analysis method

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[u(x,t)] = 0, \tag{3}$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function and x and t denote spatial and temporal independent variables, respectively.
By means of generalizing the traditional homotopy method, (see Liao [13])

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar N[\phi(x,t;p)] \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;p)$ is an unknown function. It is important to note that we have great freedom to choose auxiliary objects such as $\hbar$ and L in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t) \tag{5}$$

respectively. Thus, as p increases from 0 to1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series whit respect to p, one has

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m, \tag{6}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}|_{p=0}. \tag{7}$$

If the auxiliary linear operator, the initial guessand the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then, as proved by [14], the series (6) converges at $p=1$ and one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{8}$$

which most be one of solutions of the original nonlinear equation, as proved by [13]. As $\hbar = 1$, Eq. (4) becomes

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] + pN[\phi(x,t;p)] = 0, \tag{9}$$

which is used in the homotopy perturbation method [9].
According to the definition(7), the governing equation of can be deduced from the zero-order deformation equation (4). Define the vector

$$\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\}\,.$$

Differentiating Eq. (4) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m[\overrightarrow{u}_{m-1}(x,t)], \tag{10}$$

where

$$\Re(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}}|_{p=0}, \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases} \tag{12}$$

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by the linear equation (10) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3.The Brachistochrone Problem

One of the classical problems in calculus of variations is brachistochrone problem, that may be formulated as follows [6].

Minimize the performance index J,

$$J = \int_0^1 \left[\frac{1 + U^2(t)}{1 - X(t)}\right]^{\frac{1}{2}} dt \tag{13}$$

subject to

$$\dot{X}(t) = U(t) \tag{14}$$

with

$$X(0) = 0, \qquad X(1) = -0.5. \tag{15}$$

Equations (13),(14) and (15) describe the motion of a bead sliding down a frictionless wire in a constant gravitational field. The minimal time transfer expression (13) is obtained from the law of conservation of energy. As is well known, the exact solution to the brachistochrone problem is the cycloid defined by the parametric equations

$$X = 1 - \frac{\beta}{2}(1 + \cos 2\theta), \quad t = \frac{t_0}{2} + \frac{\beta}{2}(2\theta + \sin 2\theta),$$

where

$$\tan\theta = \dot{X}(t) = U.$$

With the given boundary conditions, the integration constants are found to be

$$\beta = 1.6184891,$$

$$t_0 = 2.7300631.$$

With attention to the general form of the calculus of variations problem, in the spatial case

$$J[y(x)] = \int_a^b f(x, y)\sqrt{1 + y'^2}dx$$

the Euler-Lagrange equation is

$$f_y - f_x y' - f\frac{y''}{1 + y'^2} = 0.$$

Thus, the Euler-Lagrange equation of the brachistochrone problem is written in the following form:

$$U' = -\frac{1}{2}\frac{1 + U^2}{X - 1}.$$

4.The homotopy analysis method for Solving the brachistochrone problem

To solve Eq. (13) by means HAM, we choose the initial approximation

$$u_0(t) = -\frac{1}{2}t. \tag{16}$$

Eq. (13) sugestes the nonlinear operator as

$$N[\phi(t;q)] = \phi(t;q)\frac{\partial^2\phi(t;q)}{\partial t^2} - \frac{\partial^2\phi(t;q)}{\partial t^2} + \frac{1}{2}(\frac{\partial\phi(t;q)}{\partial t})^2 + \frac{1}{2} \tag{17}$$

and the linear operator

$$L[\phi(t;q)] = \frac{\partial^2\phi(t;q)}{\partial t^2}, \tag{18}$$

with the property

$$L(c_1 t + c_2) = 0,$$

where c_1 and c_2 are the integration constants. Using the above definitions, we construct the zeroth-order deformation equation

$$(1-q)L[\phi(t;q) - u_0(t)] = p\hbar N[\phi(t;q)]. \tag{19}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$ and $\phi(t;q)$ is an unknown function. Obviously, when $q = 0$ and $q = 1$,

$$\phi(t;0) = u_0(t), \qquad \phi(t;1) = u(t).$$

Therefore, as the embedding parameter q increases from 0 to 1, $\phi(t;q)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Then, we obtain the mth-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = \hbar \Re_m[\overrightarrow{u}_{m-1}(t)], \tag{20}$$

subject to initial condition

$$u_m(0) = 0,$$

where

$$\Re_m(\overrightarrow{u}_{m-1}) = -\frac{\partial^2 u_{m-1}(t)}{\partial t^2} + \frac{1}{2}(1-\chi_m) + \sum_{j=0}^{m-1}[u_j(t)\frac{\partial^2 u_{m-1-j}(t)}{\partial t^2}] + \frac{1}{2}\sum_{j=0}^{m-1}[\frac{\partial u_j(t)}{\partial t}\frac{\partial u_{m-1-j}(t)}{\partial t}] \tag{21}$$

Now, the solution of the mth-order deformation equation (20) for $m \geq 1$ becomes

$$u_m(t) = \chi_m u_{m-1}(t) + \hbar L^{-1}[\Re_m(\overrightarrow{u}_{m-1})], \tag{22}$$

From (16) and (20) we now successively obtain

$$\begin{aligned}
u_0(t) &= -\tfrac{1}{2}t,\\
u_1(t) &= \tfrac{5}{16}\hbar t^2 - \tfrac{5}{16}\hbar t,\\
u_2(t) &= -\tfrac{5}{192}\hbar t(t-1)(4\hbar t + 13\hbar - 12),\\
u_3(t) &= \tfrac{5}{3072}\hbar t(t-1)(44\hbar^2 t^2 + 116\hbar^2 t - 128\hbar t - 416\hbar + 223\hbar^2 + 192),\\
u_4(t) &= -\tfrac{1}{36864}\hbar t(t-1)(1896\hbar^3 t^3 - 7920\hbar^2 t^2 + 7056\hbar^3 t^2 - 20880\hbar^2 t + 11520\hbar t\\
&\quad +9116 t\hbar^3 + 37440\hbar - 40140\hbar^2 - 11520 + 14141\hbar^3),
\end{aligned}$$

and so on. we find the approximated solution as

$$u_{app} = \sum_{i=0}^{n} u_i. \tag{23}$$

Figure 1: The $\hbar$-curve of $u_t(0,0)$ given by 6th-order HAM approximation solution.

For example for $n = 6$ we have,

$$\begin{aligned} u_{app} = &-\tfrac{1}{24772608}t(12386304 - 142541280\hbar^4 - 46448640\hbar t + 38707200\hbar^2 t^2 - 125798400\hbar^2 \\ &+59125752\hbar^5 + 52012800\hbar^4 t^3 + 20764800\hbar^4 t^2 + 87091200\hbar^2 t - 35481600\hbar^3 t^3 \\ &-58060800\hbar^3 t^2 - 18694872t\hbar^5 + 877440\hbar^6 t^6 + 3375960\hbar^6 t^5 + 4623696\hbar 6t^4 \\ &+3027360\hbar^6 t^3 - 5595716\hbar^6 t^2 + 3659019t\hbar^6 + 50652000t\hbar^4 - 6189120\hbar^5 t^5 \\ &-19656000\hbar^5 t^4 - 24020640\hbar^5 t^3 + 9434880\hbar^5 t^2 + 179827200\hbar^3 + 19111680\hbar^4 t^4 \\ &-86284800t\hbar^3 - 9967759\hbar^6 + 46448640\hbar). \end{aligned}$$

To investigate the influence of $\hbar$ on the solution series, we plot the so-called $\hbar$-curve $u_t(0,0)$ obtained from the 6th-order HAM approximation solution as shown in Fig. 1. According to this $\hbar$-curve, it is easy to discover the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis. From Fig. 1 it is clear that the series of $u_t(0,0)$ is convergent when $0.5 < \hbar < 1.5$.

In Table 1, the results for proposed method with $N = 6, 8, 11, 15$ and $\hbar = 1$ are listed. We compare the solution obtained using the proposed method with other solutions in the literature together with the exact solution. Table 2 shows the results for proposed method for different values of $\hbar$ with $N = 8$.

Table 1: The homotopy analysis method for $\hbar = 1$ and other solution in the literature

Methods	$X(1)$	$U(0)$	J
Dynamics programming Gradient method [2]	-0.5	-0.7832283	0.9984988
Dynamics programming Successive sweep method [1,3]	-0.5	-0.7834292	0.9984989
Chebyshev solution[24]			
$M = 4$	-0.5	-0.7844893	0.9984982
Legendre wavelet method[19]			
$k = N = 2, s = 5, M = 5$	-0.5	-0.7864402	0.9984981
Legendre cardinal method [7]			
$N = 6$	-0.5	-0.7863535	0.99849814831
Homotopy analysis method			
$N = 6$	-0.5	-0.78637086	0.99849817145
$N = 8$	-0.5	-0.78643798	0.99849814965
$N = 11$	-0.5	-0.78644059	0.99849814883
$N = 15$	-0.5	-0.78644078	0.99849814883
Exact solution [3]	-0.5	-0.7864408	0.99849814829

Table 2: The homotopy analysis method for different values of $\hbar$ with $N = 8$

$\hbar$	0.5	0.75	0.86	1	1.2	1.5
$U(0)$	-0.7859236	-0.7864398	-0.7864408	-0.78643798	-0.7861958	-0.7967967
J	0.99849943	0.99849963	0.99849970	0.99849814	0.99849162	1.00304829

5. Conclusion

The homotopy analysis method is considered to find the approximate solution of the brachistochrone problem. The method is easy to implement and yields very accurate results.

References

[1] A.V. Balakrishnan and L. W. Neustadt, Computing Methods in optimization Problems, Academic Preess, New York, 1964.

[2] R. Bellman, Dynamic Programming , Princeton University Press, Lawrenceville, NJ, 1957.

[3] A. E. Bryson and Y.C.Ho, Applied Optimal Control , Blaisdell Waltham, 1969.

[4] R. Y. Chang and M. L. Wang, Shifted Legendre direct method for variational problems, J. Optim. Theo. and Appl. 39 (1983), 299-307.

[5] C. F. Chen and C. H. Hsiao, Walsh series analysis in optimal control, Internat. J. of Control 21 (1975), 881-897.

[5] M. Dehghan and M. Tatari, The use of Adomian decomposition method for solving problems in calculus of variations, Mathematical Problem in Engineering (2006), 1-12.

[6] P. Dyer and S. R. Mcreynolds, The Computational and Theory of Optimal Control, Academic Press, New York, 1970.

[7] L. Elsgolts, Differential Equations and the Calculus of Variations, Mir, Moscow, translated from the Russian by G. Yankovsky, 1977.

[8] I. M. Gelfand and S. V. Fomin, Calculus of Variations, Prentice Hall, New Jersey, revised English edition translated and edited by R. A. Silverman, 1963.

[9] J. H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Engrg. 178 (1999) 257-262.

[10] I. R. Horng and J. H. Chou, Shifted Chebyshev direct method for solving variational problems, Internat. J. Systems Sci. 16 (1985), 855-861.

[11] C. Hwang and Y. P. Shih, Laquerre series direct method for variational problems, J. Optim. Theo. Appl. 39(1) (1983), 143-149.

[12] M. Lakestani, Solving brachistochrone problem using legendre cardinal functions, Int. J. Numer. Methods Appl. 4(2) (2010) 77-87. [13] S. J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton,2003.

[14] S.J. Liao, The proposed homotopy analysis method techniques for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, 1992.

[15] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499-513.

[16] M. Razzaghi and Y. Ordokhani, An application of rationalized Haar functions for variational problems, Appl. Math. Comput. 122 (2001), 353-364.

[17] M. Razzaghi and M. Razzaghi, Fourier series direct method for variational problems, Inter. J. Control. 48 (1988), no.3,887-895.

[18] M. Razzaghi and S. Yousefi, Legendre wavelets direct method for variational problems, Math. Comput. 53 (2000), 185-192.

[19] M. Razzaghi and S. Yousefi, Legendre wavelets method for the solution of nonlinear problems in the calculus of variations, Math. Comput. Modelling 34 (2001), 45-54.

[20] A. Saadatmandi and M. Dehghan, The numerical solution of problems in calculus of variation using Chebyshev finite difference method, Phys. Lett. A 372 (2008), 4037 4040.

[21] D. S. Szarkowicz, Investigating the brachistochrone with a multistage Monte Carlo method, Internat. J. Sys. Sci. 26 (1995), 233-243.

[22] M. Tatari and M. Dehghan, Solution of problems in calculus of variations via Hes variational iteration method, Phys. Lett. A 362 (2007), 401-406.

[23] V. M. Tikhomirov,Stories About Maxima and Minima, Providenc, RI,1990.

[24] J. Vlassenbroeck and R. Van Dooren, A new look at the brachistochrone problem, Appl. Math. Phy (ZAMP) 31(1980), 785-790.

Mehrdad Lakestani
Department of Applied Mathematics, Faculty of Mathematical Sciences
University of Tabriz
Tabriz-Iran
email:*lakestani@gmail.com, lakestani@tabrizu.ac.ir*

Mahnaz Kazemian
Department of Mathematics
Islamic Azad University
Tabriz-Iran
email:*kazemianmahnaz@yahoo.com*