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TOPOLOGICAL *GT*-ALGEBRAS

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Abstract. We introduce the notion of topological GT-algebras and find some properties of this structure.

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1. Introduction

The variety of Tarski algebras was introduced by J. C. Abbott in [1]. These algebras are an algebraic counterpart of the $\{\vee, \rightarrow\}$-fragment of the propositional classical calculus. S. A. Celani [3] introduced Tarski algebras with a modal operator as a generalization of the concept of the Boolean algebra with a modal operator which he researched into these fragments of the algebraic viewpoint. Kim et al. [6] established a new algebra called GT-algebra, which is a generalization of Tarski algebra, and gave a method to construct a GT-algebras from a quasi-ordered set.In [4], we introduced a topology induced by uniformity in GT-algebras and we proved that the GT-algebraic operation $\rightarrow$ is continuous with respect to this topology. Generally, in this paper some topologies are studied with a special property, which is continuity $\rightarrow$ with respect to them. We prove some properties of topological GT-algebras. We give a characterization of a topological GT-algebra in terms of neighborhoods.

2. Periliminiaries

Definition 2.1.[6] *A Generalized Tarski algebra (GT-algebra, for short) is an algebra (A, →, 1) with a binary operation →, and a constant 1 such that:*
(T1)$(\forall a \in A)$ $(1 \rightarrow a = a)$,
(T2) $(\forall a \in A)$ $(a \rightarrow a = 1)$,
(T3) $(\forall a, b, c \in A)(a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c))$.
Given a GT-algebra A, if it satisfies the condition
(T4) $(\forall a, b \in A)((a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a)$,

we call the algebra a Tarski algebra. In a Tarski algebra A we can define an order relation $\leq$ by setting $a \leq b$ if and only if $a \to b = 1$. It is well known that $(A, \leq)$ is an ordered set.[3]
A reflexive transitive relation $\Re$ on a set X is called a quasi-ordering of X and the couple $(X, \Re)$ is called a quasi-ordered set [2]. Note that If A is a GT-algebra, then the relation $\leq$ by setting $x \leq y$ if and only if $x \to y = 1$, for any $x, y \in A$ is a quasi-ordering of A; with respect to this quasi-ordering 1 is the greatest element of A [7].

Definition 2.2.[6] *A GT-filter of a GT-algebra A is a nonempty subset F of A such that for all $a, b \in A$, we have*
(F1) $b \in F \Rightarrow a \to b \in F$,
(F2) $a \to b \in F, a \in F \Rightarrow b \in F$.

Theorem 2.3.[6] *Let F be a non-empty subset of a GT-algebra A. Then F is a GT-filter of A if and only if $1 \in F$ and (F2).*

Let X be a nonempty set and U, V be any subset of $X \times X$. Define
$U \circ V = \{(x, y) \in X \times X \mid (z, y) \in U$ and $(x, z) \in V$, for some $z \in X\}$,
$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$,
$\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

Definition 2.4.[5] *By a uniformity on X we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:*
(U_1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$,
(U_2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,
(U_3) if $U \in \mathcal{K}$, then there exist a $V \in \mathcal{K}$such that $V \circ V \subseteq U$,
(U_4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,
(U_5) if $U \in \mathcal{K}$, and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a *uniform structure* (uniform space).

Theorem 2.5.[4] *Let Λ be an arbitrary family of normal GT- filters of A which is closed under intersection. If $U_F = \{(x, y) \in A \times A \mid x \equiv_F y\}$ and $\mathcal{K}^* = \{U_F \mid F \in \Lambda\}$, then $\mathcal{K}^*$ satisfies the conditions (U_1)-(U_4).*

Theorem 2.6.[4] *Let $\mathcal{K} = \{U \subseteq A \times A \mid U_F \subseteq U$ for some $U_F \in \mathcal{K}^*\}$. Then $\mathcal{K}$ satisfies a uniformity on A and the pair $(A, \mathcal{K})$ is a uniform structure.*

Let $x \in A$ and $U \in \mathcal{K}$. Define

$$U[x] := \{y \in A \mid (x, y) \in U\}.$$

Theorem 2.7.[4] *Given a GT-algebra A, then*

$$T = \{G \subseteq A \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$$

is a topology on A.

Note that for any x in A, $U[x]$ is an open neighborhood of x.

Definition 2.8.[4] *Let $(A, \mathcal{K})$ be a uniform structure. Then the topology T is called the uniform topology on A induced by $\mathcal{K}$.*

Theorem 2.9.[4] *The pair (A, T) is a topological GT-algebra, where T is uniform topology on A.*

3. Topological GT-algebra

Let A be a GT-algebra and C, D subsets of A. Then we define $C \to D$ as follows:

$$C \to D = \{x \to y \mid x \in C, y \in D\}$$

Let A be a GT- algebra and T a topology defined on the set A. Then we say that the pair (A, T) is a topological GT-algebra if the GT-algebraic operation $\to$ is continuous with respect to T. The continuity of the GT-algebraic operation $\to$ is equivalent to having the following properties satisfied:

Let O be an open set and $a, b \in A$ such that $a \to b \in O$. Then there are O_1 and O_2 such that $a \in O_1$, $b \in O_2$ and $O_1 \to O_2 \subseteq O$.

Example 3.1. Let $A = \{a, b, c, 1\}$. Define $\to$ as follow:

$\to$	a	b	c	1
a	1	b	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

Easily we can check that $(A, \rightarrow, 1)$ is a GT-algebra.

Consider

$$T = \{\{a\}, \{1, c\}, \{b\}, \{a, b\}, \{a, c, 1\}, \{a, b, c, 1\}, \emptyset\ , \{b, 1, c\}\}$$

Then (A, T) is a uniform topological space[4].
Hence by Theorem 2.9, (A, T) is a topological GT-algebra.

Definition 3.2. *A topological GT-algebera A is called discrete if every element admits a neighborhood consisting of that element only.*

Theorem 3.3. *If $\{1\}$ is an open set in a topological Tarski algebra (A, τ), then (A, τ) is discrete.*

Proof. Since $x \rightarrow x = 1 \in \{1\}$ and $\{1\}$ is an open set, there exist neighborhoods U and V of x such that $U \rightarrow V = \{1\}$. Let $W = U \bigcap V$ then $W \rightarrow W \subseteq U \rightarrow V = \{1\}$ and so $W \rightarrow W = \{1\}$. We show that $W = \{x\}$. Let $y \in W$ then $x \rightarrow y = y \rightarrow x = 1$, that is, $y = x$.

Theorem 3.4. *Let (A, τ) be a topological Hausdorff GT-algebra. Then $\{1\}$ is closed subset in A.*

Proof. Let A be Hausdorff, we show that $A - \{1\}$ is open. Let $x \in A - \{1\}$ then $x \neq 1$. Since A is Hausdorff, there exist neighborhoods U and V of $x, 1$ respectively such that $U \bigcap V = \emptyset$. Hence $1 \notin U$, that is, $U \subset A - \{1\}$. This implies $A - \{1\}$ is open, that is, $\{1\}$ is closed.

Corollary 3.5. *Let (A, τ) be a topological Tarski algebra. Then $\{1\}$ is closed in A if and only if A is Hausdorff.*

Proof. Let (A, τ) be a topological Hausdorff Tarski algebra. By Theorem 3.4, $\{1\}$ is closed subset in A. Let $\{1\}$ be closed and $x \neq y$, $x, y \in A$. Then $x \rightarrow y \neq 1$ or $y \rightarrow x \neq 1$. Let $x \rightarrow y \neq 1$, Since $A - \{1\}$ is open, there exist neighborhoods U and V of x and y such that $U \rightarrow V \subseteq A - \{1\}$. Then $U \bigcap V = \emptyset$ because if $U \bigcap V \neq \emptyset$ then there exist $x \in U \bigcap V$ and so $1 = x \rightarrow x \in U \rightarrow V$, that is, $U \rightarrow V \nsubseteq A - \{1\}$ and so this is a contradiction. Therefore A is Hausdorff.

Theorem 3.6. *Let F be an GT-filter of topological GT-algebra A. If 1 is an interior point of F, then F is open.*

Proof. Let $x \in F$, since $x \to x = 1 \in F$ and 1 is an interior point of F, there exist neighborhood U of 1 such that $x \to x = 1 \in U \subseteq F$. Then there exist neighborhoods W and W' of x such that $W \to W' \subseteq F$. Now for all $y \in W'$, $x \to y \in W \to W' \subseteq F$. Since $x \in F$ we get $y \in F$. Hence $x \in W' \subseteq F$, that is, F is open.

Theorem 3.7. *Let A be a topological GT-algebra and B an open set in A which is a subalgebra of A. Then B is a topological GT-algebra.*

Proof. We show that the *GT*-operation $\to$ is continuous in the topological space B. For all $x, y \in B$ and every neighborhood W_B of $x \to y$ in space B may be written as the follow $W_B = W \bigcap B$, for some an neighborhood W of $x \to y$ in the space A. Since A is a topological *GT*-algebra hence there exist neighborhoods U of x and V of y such that $U \to V \subseteq W$. Now let $U_B = U \bigcap B$ and $V_B = V \bigcap B$. Then U_B and V_B are neighborhoods of x and y in the topological space B. Since $U_B \to V_B = (U \bigcap B) \to (V \bigcap B) \subseteq W$ and $U_B \to V_B = (U \bigcap B) \to (V \bigcap B) \subseteq B$ then $U_B \to V_B \subseteq W \bigcap B = W_B$. Hence the operation $\to$ is continuous in the topological space B.

Theorem 3.8. *Let A be a topological GT-algebra. If F be an open subset in A which is a GT-filter then it is a closed subset in A.*

Proof. Let F be a *GT*-filter which is an open subset in A and $x \in A - F$. Since F is open and $x \to x = 1 \in F$ there exists neighborhood U such that $x \to x = 1 \in U \subseteq F$. Hence there exist neighborhoods W and W' of x such that $W \to W' \subseteq U$. Let $W_0 = W \bigcap W'$, then $W_0 \to W_0 \subseteq U$ and $x \in W_0$. We show that $W_0 \subseteq A - F$. If $W_0 \nsubseteq A - F$ then there exist $y \in W_0 \bigcap F$. We have for all $z \in W_0$, $y \to z \in W_0 \to W_0 \subseteq U \subseteq F$. Since $y \in F$ we get $z \in F$ and so $W_0 \subseteq F$. This is a contradiction.

Theorem 3.9. *Let topological GT-algebra A with the system $\{U\}$ of neighborhoods of 0 is Hausdorff, then $\bigcap U = \{0\}$.*

Proof. Let $0 \neq x \in \bigcap U$. Since A is Hausdorff there exist neighborhood U of 0 such that $x \notin U$ and so $x \notin \bigcap U$. This is a contradiction.

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