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M^K-TYPE ESTIMATES FOR MULTILINEAR COMMUTATOR OF SINGULAR INTEGRAL OPERATOR WITH GENERAL KERNEL

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Abstract. In this paper, we prove the M^k-type inequality for multilinear commutator related to generalized singular integral operator. By using the M^k-type inequality, we obtain the weighted L^p-norm inequality and the weighted estimate on the generalized Morrey spaces for the multilinear commutator.

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1. Introduction and Preliminaries

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator. Consider the commutator defined by

$$[b, T](f) = bT(f) - T(bf).$$

As the development of singular integral operators(see [5][16]), their commutators have been well studied. In [4][13][14][15], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In this paper, we will study some singular integral operators as following (see [1][8]).

Definition 1. Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y)f(y)dy$$

for every bounded and compactly supported function f, where K satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|)dx \leq C,$$

and

$$\left(\int_{2^k|z-y|\le|x-y|<2^{k+1}|z-y|}(|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|)^q dy\right)^{1/q}$$
$$\le \quad C_k(2^k|z-y|)^{-n/q'},$$

where $1<q'<2$ and $1/q+1/q'=1$.

Suppose b_j $(j=1,\cdots,m)$ are the fixed locally integrable functions on R^n. The multilinear commutator of the singular integral operator is defined by

$$T_{\vec{b}}(f)(x)=\int_{R^n}\prod_{j=1}^{m}(b_j(x)-b_j(y))K(x,y)f(y)dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies **Definition 1** with $C_j=2^{-j\delta}$(see [5][16]).

Also note that when $m=1$, $T_{\vec{b}}$ is just the commutator what we mentioned above. It is well known that multilinear operator are of great interest in harmonic analysis and have been widely studied by many authors (see [13-14]). In [15], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The purpose of this paper has two-fold, first, we establish a M^k-type estimate for the multilinear commutator related to the generalized singular integral operators, and second, we obtain the weighted L^p-norm inequality and the weighted estimates on the generalized Morrey space for the multilinear commutator by using the M^k-type inequality.

Definition 2. Let φ be a positive, increasing function on R^+ and there exists a constant $D>0$ such that

$$\varphi(2t)\le D\varphi(t) \quad \text{for} \quad t\ge 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $1\le p<\infty$,

$$||f||_{L^{p,\varphi}(w)}=\sup_{x\in R^n,\ d>0}\left(\frac{1}{\varphi(d)}\int_{Q(x,d)}|f(y)|^p w(y)dy\right)^{1/p},$$

where $Q(x,d)=\{y\in R^n:|x-y|<d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}(R^n,w)=\{f\in L^1_{loc}(R^n):||f||_{L^{p,\varphi}(w)}<\infty\}.$$

If $\varphi(d)=d^\delta$, $\delta>0$, then $L^{p,\varphi}(R^n,w)=L^{p,\delta}(R^n,w)$, which is the classical Morrey spaces (see [11][12]). If $\varphi(d)=1$, then $L^{p,\varphi}(R^n,w)=L^p(w)$, which is the weighted Lebesgue spaces (see [5]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [3][6][7][9][10]).

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$(f)^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [5][16])

$$(f)^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$.

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $0 < p < \infty$, we denote $M_p f(x)$ by

$$M_p(f)(x) = [M(|f|^p)(x)]^{1/p}.$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e. $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \qquad when \;\; k \geq 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|fy|}{\lambda} \right) d(y) \leq 1 \right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x \in Q} ||f||_{\Phi,Q}.$$

The Young functions to be using in this paper are $\Phi(t) = t(1 + logt)^r$ and $\tilde{\Phi}(t) = exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL)^r,B}$,

$M_{L(logL)^r}$ and $||\cdot||_{expL^{1/r},B}$, $M_{expL^{1/r}}$. Following [13][14], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \le ||f||_{\Phi,Q}||g||_{\tilde{\Phi},Q}.$$

And we can also obtain the following inequalities:

$$||f||_{L(logL)^{1/r},Q} \le M_{L(logL)^{1/r}}(f) \le CM_{L(logL)^m}(f) \le CM^{m+1}(f),$$

$$||b-b_Q||_{expL^r,Q} \le C||b||_{BMO},$$

$$|b_{2^{k+1}Q}-b_{2Q}| \le Ck||b||_{BMO}.$$

for $r, r_j \ge 1, j=1,2,\cdots,m$ with $1/r = 1/r_1 + 1/r_2 \cdots + 1/r_m$, and $b \in BMO(R^n)$.

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements and $\sigma(i) < \sigma(j)$ when $i < j$. For $\sigma \in C_j^m$, set $\sigma^c = \{1,\cdots,m\} \setminus \sigma$. For $\vec{b} = (b_1,\cdots,b_m)$ and $\sigma = \{\sigma(1),\cdots,\sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)},\cdots,b_{\sigma(j)})$, $b_\sigma = \prod_{i=1}^{j} b_{\sigma(i)}$ and $||\vec{b}_\sigma||_{BMO} = \prod_{i=1}^{j} ||b_{\sigma(i)}||_{BMO}$.

We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$(see [5]), that is

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}$$

and

$$A_p = \left\{w : \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty\right\},\ 1 < p < \infty.$$

2. Theorems and Proofs

Now we give some theorems as following.

Theorem 1. *Let T be the singular integral operator as* **Definition 1**, *the sequence $\{k^m C_k\} \in l^1$, $q' \le s < \infty$, $0 < r < 1$, $k \ge m+1$, $k \in N$ and $b_j \in BMO(R^n)$ for $j = 1,\cdots,m$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,*

$$(T_{\vec{b}}(f))_r^{\#}(\tilde{x}) \le C||\vec{b}||_{BMO}\left(M^k(f)(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m} M^k(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}) + M_s(f)(\tilde{x})\right).$$

Theorem 2. *Let* T *be the singular integral operator as* **Definition 1**, *the sequence* $\{k^m C_k\} \in l^1$, $q' \le p < \infty$, $w \in A_p$ *and* $b_j \in BMO(R^n)$ *for* $j = 1, \cdots, m$. *Then* $T_{\vec{b}}$ *is bounded on* $L^p(w)$.

Theorem 3. *Let* T *be the singular integral operator as* **Definition 1**, *the sequence* $\{k^m C_k\} \in l^1$, $q' \le p < \infty$, $w \in A_1$ *and* $b_j \in BMO(R^n)$ *for* $j = 1, \cdots, m$. *Then, if* $0 < D < 2^n$,

$$||T_{\vec{b}}(f)||_{L^{p,\varphi}(w)} \le C||\vec{b}||_{BMO}||f||_{L^{p,\varphi}(w)}.$$

In order to better proof of the theorem above, we need the following lemmas

Lemma 1. *Let* $1 < r < \infty$ *and* $b_j \in BMO(R^n)$ *with* $j = 1, \cdots, k$ *and* $k \in N$. *Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^{k} ||b_j||_{BMO},$$

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \le C \prod_{j=1}^{k} ||b_j||_{BMO}.$$

Similarly, for $\sigma \in C_k^m$, *when* $k \le m$ *and* $m \in N$, *we have:*

$$\frac{1}{|Q|}\int_Q |(b(y) - (b_j)_Q)_\sigma| dy \le C||b_\sigma||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q |(b(y) - (b_j)_Q)_\sigma|^r dy\right)^{1/r} \le C||b_\sigma||_{BMO}.$$

In fact, we just need to choose $p_j > 1$ and $q_j > 1$, where $1 \le j \le k$, such that $1/p_1 + \cdots + 1/p_k = 1$ and $r/q_1 + \cdots + r/q_k = 1$. After that, using the Hölder's inequality with exponent $1/p_1 + \cdots + 1/p_k = 1$ and $r/q_1 + \cdots + r/q_k = 1$. respectively, we may get the results.

Lemma 2. ([5, p.485]) *Let* $0 < p < q < \infty$ *and for any function* $f \ge 0$. *We define that, for* $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E *with* $0 < |E| < \infty$. *Then*

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p}||f||_{WL^q}.$$

Lemma 3.(see [5])*Let* $0 < p, \eta < \infty$ *and* $w \in \cup_{1\le r<\infty} A_r$. *Then*

$$||M_\eta(f)||_{L^p(w)} \le C||f_\eta^{\#}(f)||_{L^p(w)}.$$

Lemma 4.*Let* $1 < p < \infty$, $1 \le q < p$ *and* $w \in A_1$. *Then, if* $0 < D < 2^n$,

$$||M_q(f)||_{L^{p,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}.$$

Proof. Let $f \in L^{p,\varphi}(R^n, w)$. Note that $1 \le q < p$ and for any $w \in A_1$,

$$\int_{R^n} |M_q(f)(y)|^p w(y)dy \le C\int_{R^n} |f(y)|^p w(y)dy.$$

For a cube $Q = Q(x, d) \subset R^n$, we get

$$\begin{aligned}
& \int_Q |M_q(f)(y)|^p w(y)dy \\
\le\ & \int_{R^n} |M_q(f)(y)|^p M(w\chi_Q)(y)dy \\
\le\ & C\int_{R^n} |f(y)|^p M(w\chi_Q)(y)dy \\
=\ & C\left[\int_Q |f(y)|^p M(w\chi_Q)(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} |f(y)|^p M(w\chi_Q)(y)dy\right] \\
\le\ & C\left[\int_Q |f(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} |f(y)|^p \frac{w(y)}{|2^{k+1}Q|}dy\right] \\
\le\ & C\left[\int_Q |f(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q} |f(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}dy\right] \\
\le\ & C\left[\int_Q |f(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q} |f(y)|^p \frac{w(y)}{2^{nk}}dy\right] \\
\le\ & C||f||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty} 2^{-nk}\varphi(2^{k+1}d) \\
\le\ & C||f||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty} (2^{-n}D)^k\varphi(d) \\
\le\ & C||f||^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus

$$||M_q(f)||_{L^{p,\varphi}(\omega)} \le C||f||_{L^{p,\varphi}(w)}.$$

Lemma 5. *Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$. Then, for $f \in L^{p,\varphi}(R^n, w)$,*

$$||M(f)||_{L^{p,\varphi}(w)} \le C||f^{\#}||_{L^{p,\varphi}(w)}.$$

Lemma 6. *Let T be the bounded linear operators on $L^q(R^n, w)$ for any $1 < q < \infty$ and $w \in A_1$. Then, for $1 < p < \infty$, $w \in A_1$ and $0 < D < 2^n$,*

$$||T(f)||_{L^{p,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}.$$

The proofs of two Lemmas are similar to that of Lemma 4, we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x) - C_0|^r dx\right)^{1/r} \le C||\vec{b}||_{BMO}\left(M^k(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma\in C_j^m} M^k(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x})\right).$$

Fix a ball $Q = Q(x_0, d)$ and $\tilde{x} \in Q$, we write $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$. Following [20], we will consider the cases $m = 1$ and $m > 1$, and choose $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$ and $C_0 = T(\prod_{j=1}^m (b_j - (b_j)_{2Q})f_2)(x_0)$, respectively.

We first consider the **Case** $m = 1$. For $C_0 = T(((b_1)_{2Q} - b_1)f_2)(x_0)$, we write

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})T(f)(x) - T((b_1 - (b_1)_{2Q})f)(x).$$

Then

$$\begin{aligned}
& |T_{b_1}(f)(x) - C_0| \\
= & |(b_1(x) - (b_1)_{2Q})T(f)(x) + T(((b_1)_{2Q} - b_1)f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
\le & |(b_1(x) - (b_1)_{2Q})T(f)(x)| + |T(((b_1)_{2Q} - b_1)f_1)(x)| \\
& + |T(((b_1)_{2Q} - b_1)f_2)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
= & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, we get

$$\begin{aligned}
& \left(\frac{1}{|Q|}\int_Q |A(x)|^r dx\right)^{1/r} \\
\le & \frac{1}{|Q|}\int_Q |A(x)| dx \\
\le & \frac{1}{|Q|}\int_Q |(b_1(x) - (b_1)_{2Q})T(f)(x)| dx \\
\le & ||b_1 - (b_1)_{2Q}||_{\exp L, 2Q}||T(f)||_{L(\log L), 2Q} \\
\le & C||b_1||_{BMO} M^2(T(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, by the weak type (1,1) of T and Lemma 2 , we obtain

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |B(x)|^r dx\right)^{1/r}\\
\le\ & \frac{1}{|Q|}\int_Q |B(x)|dx\\
=\ & \frac{1}{|Q|}\int_Q |T(((b_1)_{2Q}-b_1)f_1)(x)|dx\\
\le\ & \left(\frac{1}{|Q|}\int_{2Q} |T((b_1-(b_1)_{2Q})f\chi_{2Q})(x)|^p dx\right)^{1/p}\\
=\ & \frac{1}{|Q|}\frac{1}{|Q|^{\frac{1}{p}-1}}||T((b_1-(b_1)_{2Q})f\chi_{2Q})||_{L^p}\\
\le\ & \frac{C}{|Q|}||T((b_1-(b_1)_{2Q})f\chi_{2Q})||_{WL^1}\\
\le\ & \frac{C}{|Q|}||((b_1-(b_1)_{2Q})f\chi_{2Q}||_{L^1}\\
\le\ & \frac{C}{|Q|}\int_{2Q} |b_1(x)-(b_1)_{2Q}||f(x)|dx\\
\le\ & C||b_1-(b_1)_{2Q}||_{expL,2Q}||f||_{L(\log L),2Q}\\
\le\ & C||b_1||_{BMO}M^2(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, recalling that $s>q'$, taking $1<p<\infty, 1<t<s$ with $1/p+1/q+1/t=1$, by the Hölder's inequality, we have, for $x\in Q$,

$$\begin{aligned}
&|T((b_1-(b_1)_{2Q})f_2)(x)-T((b_1-(b_1)_{2Q})f_2)(x_0)|\\
=\ & \left|\int_{(2Q)^c}(b_1(y)-(b_1)_{2Q})f(y)(K(x,y)-K(x_0,y))dy\right|\\
\le\ & \sum_{k=1}^{\infty}\int_{2^k|x-x_0|\le|y-x_0|<2^{k+1}|x-x_0|}|K(x,y)-K(x_0,y)||f(y)||b_1(y)-(b_1)_{2Q}|dy\\
\le\ & C\sum_{k=1}^{\infty}\left(\int_{2^k|x-x_0|\le|y-x_0|<2^{k+1}|x-x_0|}|K(x,y)-K(x_0,y)|^q dy\right)^{1/q}\\
&\times\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|b_1(y)-(b_1)_{2Q}|^p dy\right)^{1/p}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^t dy\right)^{1/t}\\
\le\ & C\sum_{k=1}^{\infty}C_k\frac{|2^{k+1}Q|^{1/p+1/t}}{(2^kd)^{n/q'}}k||b_1||_{BMO}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s}
\end{aligned}$$

$$\begin{aligned}
&\leq C||b_1||_{BMO}\sum_{k=1}^{\infty}kC_kM_s(f)(\tilde{x})\\
&\leq C||b_1||_{BMO}M_s(f)(\tilde{x}),
\end{aligned}$$

thus

$$\left(\frac{1}{|Q|}\int_Q|C(x)|^rdx\right)^{1/r}\leq C||b_1||_{BMO}M_s(f)(\tilde{x}).$$

Now, we consider the **Case** $m\geq 2$. we have, for $b=(b_1,\cdots,b_m)$,

$$\begin{aligned}
&T_{\vec{b}}(f)(x)=\int_{R^n}\prod_{j=1}^m(b_j(x)-b_j(y))K(x,y)f(y)dy\\
=&\int_{R^n}\prod_{j=1}^m[(b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q})]K(x,y)f(y)dy\\
=&\sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}K(x,y)f(y)dy\\
=&\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})\int_{R^n}K(x,y)f(y)dy+(-1)^m\int_{R^n}\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})K(x,y)f(y)dy\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b_j(x)-(b_j)_{2Q})_\sigma\int_{R^n}(b_j(y)-(b_j)_{2Q})_{\sigma^c}K(x,y)f(y)dy\\
=&\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})T(f)(x)+(-1)^mT(\prod_{j=1}^m(b_j-(b_j)_{2Q})f)(x)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}((b_j(x)-(b_j)_{2B})_\sigma T(b_j-(b_j)_{2B})_{\sigma^c}(f)(x)
\end{aligned}$$

thus, recall that $C_0=T(\prod_{j=1}^m(b_j-(b_j)_{2B})f_2)(x_0)$,

$$\begin{aligned}
&|T_{\vec{b}}(f)(x)-T(\prod_{j=1}^m(b_j-(b_j)_{2B})f_2)(x_0)|\\
\leq&|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})T(f)(x)|\\
&+|T(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_1)(x)|\\
&+|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}((b_j(x)-(b_j)_{2Q})_\sigma T(b_j-(b_j)_{2Q})_{\sigma^c}(f)(x)|
\end{aligned}$$

$$
\begin{aligned}
& +|T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x)-T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x_0)| \\
= & \ I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x)$, we get,

$$
\begin{aligned}
& \left(\frac{1}{|Q|}\int_Q |I_1(x)|^r dx\right)^{1/r} \leq \frac{1}{|Q|}\int_Q |I_1(x)|dx \\
\leq & \ \frac{1}{|Q|}\int_Q |\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})||T(f)(x)|dx \\
\leq & \ C\prod_{j=1}^{m}||(b_j-(b_j)_{2Q})||_{\exp L^{1/r_j},2Q}||T(f)||_{L(\log L)^r,2Q} \\
\leq & \ C\prod_{j=1}^{m}||b_j||_{BMO}M^{m+1}(T(f))(\tilde{x}) \\
\leq & \ C||\vec{b}||_{BMO}M^k(T(f))(\tilde{x}).
\end{aligned}
$$

For $I_2(x)$, by the boundness of T on $L^p(R^n)$ and similar to the proof of $B(x)$, using Lemma 2, we get

$$
\begin{aligned}
& \left(\frac{1}{|Q|}\int_Q |I_2(x)|^r dx\right)^{1/r} \leq \frac{1}{|Q|}\int_Q |I_2(x)|dx \\
= & \ \frac{1}{|Q|}\int_Q |T(\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_1)(x)|dx \\
\leq & \ \left(\frac{1}{|Q|}\int_Q |T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_1)(x)|^p dx\right)^{1/p} \\
= & \ \frac{1}{|Q|}\frac{1}{|Q|^{\frac{1}{p}-1}}||T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_1)||_{L^p} \\
\leq & \ \frac{1}{|Q|}||T(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_1)||_{WL^1} \\
\leq & \ \frac{1}{|Q|}||(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_1)||_{L^1} \\
\leq & \ \frac{1}{|Q|}\int_B |\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})||f_1(x)|dx
\end{aligned}
$$

$$\leq C\prod_{j=1}^{m}||(b_j-(b_j)_{2Q})||_{\exp L^{1/r_j},2Q}||f||_{L(\log L)^r,2Q}$$
$$\leq C||\vec{b}||_{BMO}M^{m+1}(f)(\tilde{x})$$
$$\leq C||\vec{b}||_{BMO}M^{k}(f)(\tilde{x}).$$

For $I_3(x)$, by Lemma 2,

$$\left(\frac{1}{|Q|}\int_Q|I_3(x)|^r d\mu(x)\right)^{1/r}\leq\frac{1}{|Q|}\int_Q|I_3(x)|dx$$
$$\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q|(b_j(x)-(b_j)_{2Q})_\sigma||T(b_j-(b_j)_{2Q})_{\sigma^c}(f)(x)|dx$$
$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b_j(x)-(b_j)_{2Q})_\sigma||_{\exp L^{1/r_j},2Q}||T(b_j-(b_j)_{2Q})_{\sigma^c}(f)||_{L(\log L)^r,2Q}$$
$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||b_\sigma||_{BMO}M^{m+1}(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x})$$
$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}||_{BMO}M^{k}(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}).$$

For $I_4(x)$, similar to the proof of $C(x)$ in the **Case** $m=1$, for $1<p<\infty, 1<t<s$ with $1/p+1/q+1/t=1$, we have

$$|T((\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x)-T((\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x_0)|$$
$$\leq C\sum_{k=1}^{\infty}\int_{2^k|x-x_0|\leq|y-x_0|<2^{k+1}|x-x_0|}|(K(x,y)-K(x_0,y))||f(y)||\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})|dy$$
$$\leq C\sum_{k=1}^{\infty}\left(\int_{2^k|x-x_0|\leq|y-x_0|<2^{k+1}|x-x_0|}|K(x,y)-K(x_0,y)|^q dy\right)^{1/q}$$
$$\times\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q}|^p dy\right)^{1/p}\left(\int_{|y-x_0|<2^{k+1}|x-x_0|}|f(y)|^t dy\right)^{1/t}$$
$$\leq C\sum_{k=1}^{\infty}C_k\frac{|2^{k+1}Q|^{1/p+1/t}}{(2^kd)^{n/q'}}k^m\prod_{j=1}^{m}||b_j||_{BMO}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^s dy\right)^{1/s}$$
$$\leq C||\vec{b}||_{BMO}\sum_{k=1}^{\infty}k^mC_kM_s(f)(\tilde{x})$$

$$\leq C||\vec{b}||_{BMO}M_s(f)(\tilde{x}),$$

thus

$$\left(\frac{1}{|Q|}\int_Q |I_4(x)|^r dx\right)^{1/r} \leq ||\vec{b}||_{BMO}M_s(f)(\tilde{x}).$$

This completes the proof of the theorem.

Proof of Theorem 2. Choose $q' < s < p$ in Theorem 1, by the $L^p(w)$-boundedness of M^k and M_s, we may obtain the conclusion of Theorem 2 by induction.

Proof of Theorem 3. We first consider the case m=1. Choose $q' < s < p$ in Theorem 1, by Theorem 1 and Lemma 4-6, we obtain

$$\begin{aligned} & ||T_{\vec{b}}(f)||_{L^{p,\varphi}(w)} \leq ||M(T_{\vec{b}}(f))||_{L^{p,\varphi}(w)} \leq C||(T_{\vec{b}})^{\#}_r(f)||_{L^{p,\varphi}(w)} \\ \leq & C||\vec{b}||_{BMO}\left(||M^k(f)||_{L^{p,\varphi}(w)} + ||M^k(T(f))||_{L^{p,\varphi}(w)} + ||M_s(f)||_{L^{p,\varphi}(w)}\right) \\ \leq & C||\vec{b}||_{BMO}\left(||f||_{L^{p,\varphi}(w)} + ||T(f))||_{L^{p,\varphi}(w)} + ||f||_{L^{p,\varphi}(w)}\right) \\ \leq & C||\vec{b}||_{BMO}\left(||f||_{L^{p,\varphi}(w)} + ||f||_{L^{p,\varphi}(w)}\right) \\ \leq & C||\vec{b}||_{BMO}||f||_{L^{p,\varphi}(w)}. \end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem 3 by induction.

This completes the proof of Theorem 3.

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