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GENERALIZED PESCAR COMPLEX NUMBERS AND OPERATIONS WITH THESE NUMBERS

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Abstract. In this paper, the author introduced the generalized Pescar complex numbers. For these new numbers, are presented operations with generalized complex numbers, the module of a generalized complex numbers, the properties of generalized complex numbers $(C_{G_x}\backslash\{0\}, \times)$, the trigonometric form of these numbers as well. On introducing the generalized complex number, the author defines the general complex numbers.
The presented findings are the results of the author's original research.

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1. Definition of generalized Pescar complex numbers

Let V_3 be the free vector space and E_3 the Euclidean real punctual space. We denote by $\dim V_3$ the vector space dimension V_3 and by $\dim E_3$, the dimension of the Euclidean real punctual space E_3.We have $\dim V_3 = \dim E_3$.
We assume that R^3 is the three-dimensional vector space over the real numbers and C the set of complex numbers, $C = \{z = x + iy \mid x, y \in R,\ i^2 = -1.\}$
Since V_3 is a three-dimensional vector space, we define an orthonormal Cartesian frame $\Re_3^{ON_1} = \{O, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$, where O is the origin of the Euclidean real punctual space E_3 and $B_{ON_1} = \{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ is an orthonormal basis of the vector space V_3.

Any free vector of V_3 can be specified through a class representative free vector with its origin in O, determining a position vector $\overrightarrow{v}$ with its origin in O, its extremity in any point M, denoted $\overrightarrow{OM}$ and having a unique decomposition with respect to basis B_{ON_1}.

$$\overrightarrow{v} = \overrightarrow{OM} = x\overrightarrow{e_1} + y\overrightarrow{e_2} + t\overrightarrow{e_3},\ \ (x, y, t) \in R^3. \tag{1}$$

The size of the vector $\overrightarrow{OM}$ is given by

$$\parallel \overrightarrow{OM} \parallel = \sqrt{x^2 + y^2 + t^2}. \tag{2}$$

If point M is on a sphere of radius r, then

$$\parallel \overrightarrow{OM} \parallel = r, \quad (r > 0, \ r = \text{constant}) \tag{3}$$

and the sphere centered at O of radius r is described by the equation

$$(\Sigma_r) : \ x^2 + y^2 + t^2 - r^2 = 0. \tag{4}$$

If M is any point, then this point is found on the spheres of variable radius $\lambda, \ \lambda > 0$. We have $\parallel \overrightarrow{OM} \parallel = \lambda$ and the equation of the spheres of variable radius λ, centered at O is

$$(\Sigma_\lambda) : \ x^2 + y^2 + t^2 - \lambda^2 = 0. \tag{5}$$

Let $Oxyt$ be the orthogonal axes system corresponding to the orthonormal Cartesian frame $\Re_3^{ON_1}$. We assume axis Ox, the real axis with the real unity 1, and axes Oy, Ot the imaginary axes with the imaginary units i, respectively j, $i^2 = j^2 = -1$. We denote this orthogonal axes system $(Oxyt)_x$, thus specifying the real axis of the axes system.

Definition 1.1. *The product of complex numbers x, a, b, ai, bj with $x, a, b \in R$, denoted by "$\times$" is defined by*

$$\begin{aligned}
x \times x = (1 \cdot x) \cdot (1 \cdot x) &= x^2, \\
x \times a = (1 \cdot x) \cdot (1 \cdot a) &= xa, \\
x \times b = (1 \cdot x) \cdot (1 \cdot b) &= xb. \\
ai \times ai = ai \cdot ai \cos 0^0 &= a\, ai^2 = -a^2, \\
bj \times bj = bj \cdot bj \cos 0^0 &= b\, bj^2 = -b^2, \\
ai \times bj = ai \cdot bj \cos 90^0 &= abij \cos 90^0 = 0, \\
x \times ai = (1 \cdot x) \cdot (ai) &= xai, \\
x \times bj = (1 \cdot x) \cdot (bj) &= xbj.
\end{aligned} \tag{6}$$

From (6), for $a = b = 1$, we obtain

$$i \times i = i \cdot i \cos 0^0 = i^2 = -1,$$

$$j \times j = j \cdot j \cos 0^0 = j^2 = -1,$$
$$i \times j = i \cdot j \cos 90^0 = 0.$$

The correspondent with respect to the axes system $(Oxyt)_x$ of the position vector $\overrightarrow{OM}$ referred to the frame $\Re_3^{ON_1} = \{O, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ is a generalized complex number with the image in point $M \in (\Sigma_\lambda)$, which has the Cartesian coordinates $(x, y, t) \in R^3$ with respect to the Cartesian axes system.

Definition 1.2. *We designate as generalized Pescar complex number, with the real part x and the imaginary parts y, t, the number of the form*

$$w = 1 \cdot x + iy + jt, \;\; (x, y, t) \in R^3, \tag{7}$$

referred to the axes system $(Oxyt)_x$.

The projections of the generalized complex number w defined by (7), on the coordinate axes Ox, Oy, Ot are denoted by

$$x = \mathrm{Re}w, \; y = \mathrm{Im}_1 w, \; t = \mathrm{Im}_2 w, \tag{8}$$

The generalized complex number w, with the real part x can be denoted by

$$w = \mathrm{Re}w + iIm_1 w + j\mathrm{Im}_2 w. \tag{9}$$

We denote the set of generalized Pescar complex numbers, with the real part x, by

$$C_{G_x} = \left\{ w \mid w = x + iy + jt, (x, \, y, \, t) \in R^3, \, i^2 = j^2 = -1 \right\}. \tag{10}$$

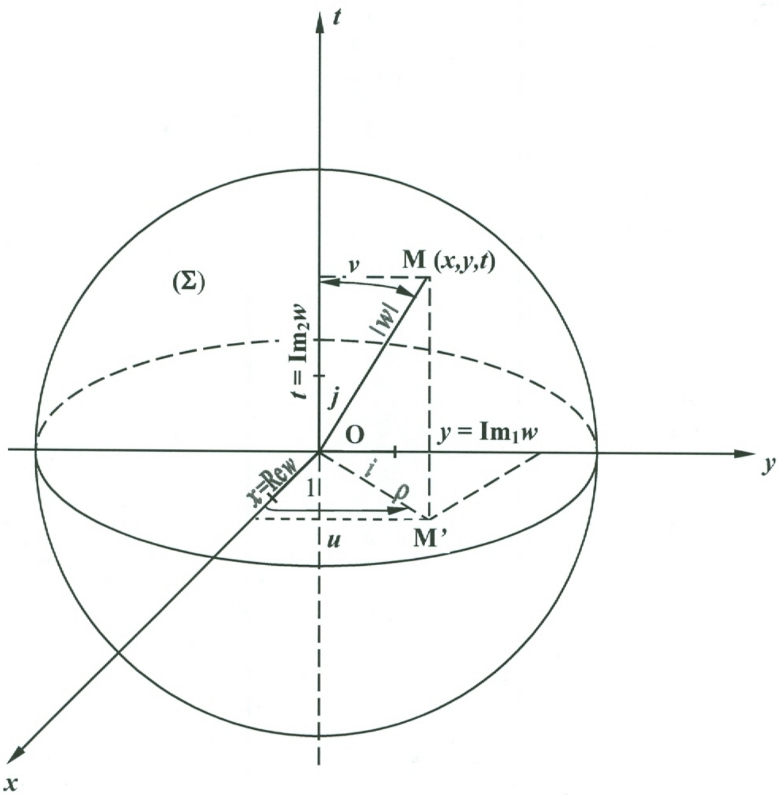

Fig. 1.1

Consequences.

p_1). The generalized complex number of the form $w_0 = 0 + i \cdot 0 + j \cdot 0$, $(0, 0, 0) \in R^3$, is the generalized null complex number corresponding to the set C_{G_x}.

p_2). The image of the null complex number w_0 with respect to the axes system $(Oxyt)_x$ is the complex point O.

p_3). The images of the complex numbers $w \in C_{G_x} \setminus \{w_0\}$ belong to the variable spheres centered in the complex point O.

p_4). The sphere centered in O is called complex sphere centered in complex point O.

p_5). For $t = 0$, we obtain the complex number $w = x + iy$, with the image in the complex plane (Oxy), $i^2 = -1,\ w \in C$.

p_6). For $y = 0$, we obtain the complex number $w = x + jt$, with the image in the complex plane (Oxt), $j^2 = -1$.

p_7). For $x = 0$, we obtain the imaginary generalized complex number $w = iy + jt,\ i^2 = j^2 = -1$, with the image in the imaginary complex plane (Oyt).

Consider the orthogonal Cartesian axes system $Oytx$, that corresponds to the orthonormal Cartesian frame $\Re_3^{ON_2} = \{O, e_2, e_3, e_1\}$ and let Oy be the real axis, with the real unity 1, and Ot, Ox, the imaginary axes, with the imaginary units j, k respectively so that $j^2 = k^2 = -1$. We denote by $(Oytx)_y$ this orthogonal axes system, thus specifying the real axis of the axes system.

Definition 1.3. *The product of complex numbers y, a, b, aj, bk with $y, a, b \in R$, denoted by "$\times$" is defined by*

$$\begin{aligned}
y \times y = (1 \cdot y)(1 \cdot y) &= y^2, \\
y \times a = (1 \cdot y)(1 \cdot a) &= ya, \\
y \times b = (1 \cdot y)(1 \cdot b) &= yb, \\
y \times aj = (1 \cdot y)(aj) &= yaj, \\
y \times bk = (1 \cdot y)(bk) &= ybk, \\
aj \times aj = aj \cdot aj \cos 0^0 &= a \cdot a \cdot j^2 = -a^2, \\
bk \times bk = bk \cdot bk \cos 0^0 &= b \cdot bk^2 = -b^2, \\
aj \times bk = aj \cdot bk \cos 90^0 &= abjk \cos 90^0 = 0, \\
aj \times bj = aj \cdot bj \cos 0^0 &= abj^2 = -ab, \\
ak \times bk = ak \cdot bkcos0^0 &= abk^2 = -ab.
\end{aligned} \tag{11}$$

From (11), for $a = b = 1$, we obtain

$$j \times j = j \cdot j \cos 0^0 = j^2 = -1,$$

$$k \times k = k \cdot k \cos 0^0 = k^2 = -1,$$

$$j \times k = j \cdot k \cos 90^0 = 0.$$

The correspondent with respect to the axes system $(Oytx)_y$ of the position vector $\overrightarrow{OM}$ referred to the frame $\Re_3^{ON_2} = \{O, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_1}\}$ is a generalized complex number with the image in point $M \in (\Sigma_\lambda)$, which has the Cartesian coordinates $(y, t, x) \in R^3$ with respect to the Cartesian axes system.

Definition 1.4. *We designate as generalized Pescar complex number, with the real part y and the imaginary parts t, x, the number of the form*

$$w = 1 \cdot y + jt + kx, \ \ (y, t, x) \in R^3, \tag{12}$$

referred to the axes system $(Oytx)_y$.

The projections of the generalized complex number w defined by (12), on the coordinate axes Oy, Ot, Ox are denoted by

$$y = \mathrm{Re}w, \ t = \mathrm{Im}_1 w, \ x = \mathrm{Im}_2 w. \tag{13}$$

The generalized complex number with the real part y can be denoted by

$$w = \mathrm{Re}w + jIm_1 w + k\mathrm{Im}_2 w. \tag{14}$$

We denote the set of generalized Pescar complex numbers, with the real part y, by

$$C_{G_y} = \left\{ w \ | \ w = y + jt + kx, (y, \, t, \, x) \in R^3, \, i^2 = j^2 = -1 \right\}. \tag{15}$$

Consequences.

m_1). The generalized complex number of the form $w_0 = 0 + 0j + 0k$, $(0, 0, 0) \in R^3$, is the generalized null complex number corresponding to the set C_{G_y}.

m_2). The image of the null complex number $w_0 \in C_{G_y}$ with respect to the axes system $(Oytx)_y$ is the complex point O.

m_3). The images of the complex numbers $w \in C_{G_y} \setminus \{w_0\}$ belong to the variable spheres, referred to $(Oytx)_y$ and centered in the complex point O.

m_4). For $x = 0$, we obtain the complex number $w = y + jt$, $j^2 = -1$, with image in the complex plane (Oyt).

m_5). For $t = 0$, we obtain the complex number $w = y + kx$, $k^2 = -1$, with image in the complex plane (Oyx).

m_6). For $y = 0$, we have the imaginary generalized complex number $w = jt + kx$, $j^2 = k^2 = -1$, with image in the imaginary complex plane (Otx).

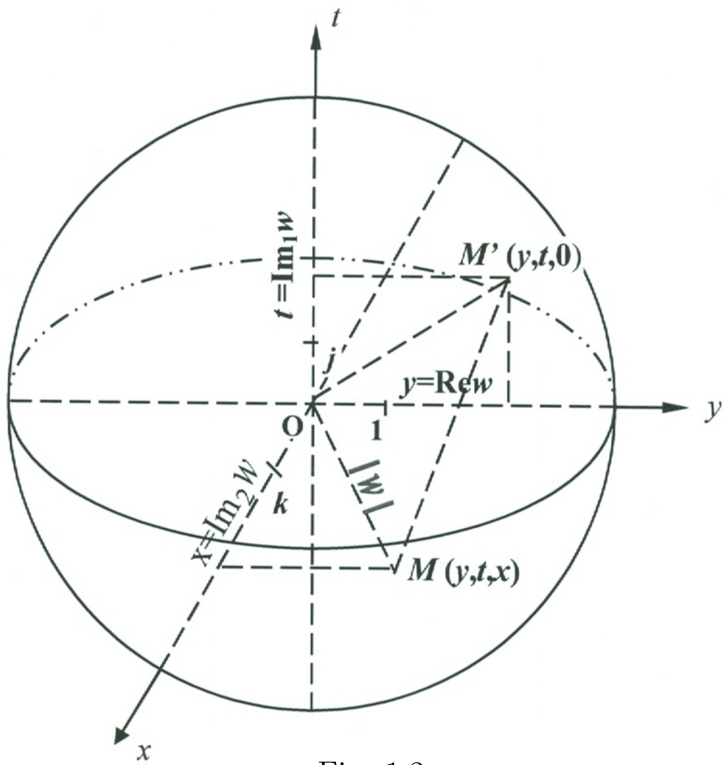

Fig. 1.2

Consider the orthogonal Cartesian axes system $Otxy$, that corresponds to the orthonormal frame $\Re_3^{ON_3} = \{O, \overrightarrow{e_3}, \overrightarrow{e_1}, \overrightarrow{e_2}\}$, and let Ot be the real axis, with the real unity 1, and Ox, Oy, the imaginary axes with the imaginary units k, i respectively so that $k^2 = i^2 = -1$.

We denote by $(Otxy)_t$ this orthogonal axes system, thus specifying the real axis of the axes system.

Definition 1.5. *The product of complex numbers t, a, b, ak, bi with $t, a, b \in R$,* *denoted by "×" is defined by*

$$\begin{aligned} t\times a=(1\cdot t)\cdot(1\cdot a) &= ta, \\ t\times b=(1\cdot t)\cdot(1\cdot b) &= tb, \\ t\times ak=(1\cdot t)\cdot(ak) &= tak, \\ t\times bi=(1\cdot t)\cdot(bi) &= tbi, \\ ak\times ak=ak\cdot ak\cos 0^0 &= aak^2=-a^2, \\ bi\times bi=bi\cdot bi\cos 0^0 &= bbi^2=-b^2, \\ ak\times bi=ak\cdot bi\cos 90^0 &= abki\cos 90^0=0. \end{aligned} \tag{16}$$

For $a=b=1$, from (16) we obtain

$$k\times k=k\cdot k\cos 0^0=k^2=-1,$$

$$i\times i=i\cdot i\cos 0^0=i^2=-1,$$

$$k\times i=k\cdot i\cos 90^0=0.$$

The correspondent with respect to the axes system $(Otxy)_t$ of the position vector $\overrightarrow{OM}$ referred to the frame $\{O,\ \overrightarrow{e_3},\ \overrightarrow{e_1},\ \overrightarrow{e_2}\}$ is a generalized complex number with the image in point $M\in(\Sigma_\lambda)$, which has the Cartesian coordinates $(t,x,y)\in R^3$ with respect to the Cartesian axes system.

Definition 1.6. *We designate as generalized Pescar complex number, with the real part t and the imaginary parts x,y, the number of the form*

$$w=1\cdot t+kx+iy,\ \ (t,x,y)\in R^3, \tag{17}$$

it referred to the axes system $(Otxy)_t$.

The projections of the generalized complex number w defined by (17), on the coordinate axes Ot, Ox, Oy are denoted by

$$t=\mathrm{Re}w,\ x=\mathrm{Im}_1 w,\ y=\mathrm{Im}_2 w,\ (t,x,y)\in R^3, \tag{18}$$

The generalized complex number, with t as the real part can be written as

$$w=\mathrm{Re}w+kIm_1 w+i\mathrm{Im}_2 w. \tag{19}$$

We denote the set of generalized Pescar complex numbers with the real part t, by

$$C_{G_t}=\left\{w\ \mid\ w=t+xk+yi,\ (t,\ x,\ y)\in R^3,\ k^2=i^2=-1\right\}. \tag{20}$$

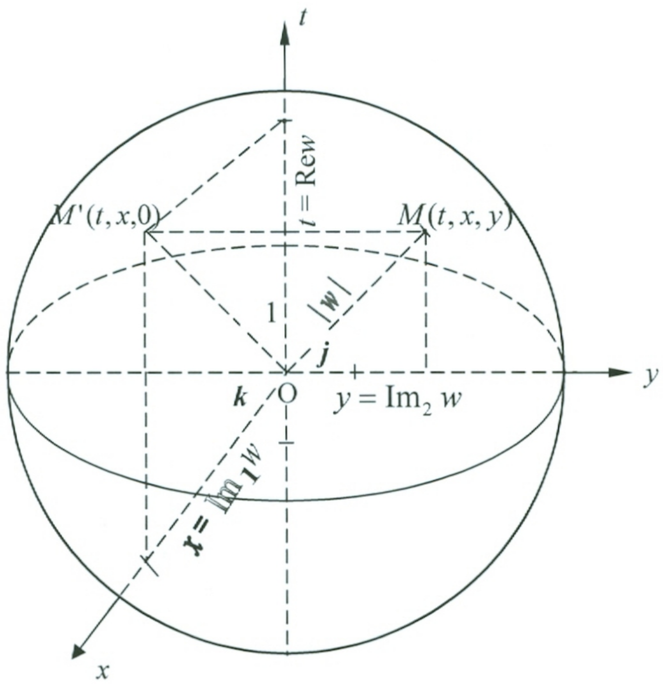

Fig. 1.3.

Consequences.

q_1). The generalized complex number of the form $w_0 = 0+0k+0i$, $(0,\ 0,\ 0) \in R^3$ is the generalized null complex number corresponding to the set C_{G_t}.

q_2). The image of the complex number $w_0 \in C_{G_t}$ with respect to the axes system $(Otxy)_t$ is the complex point O.

q_3). The images of the complex numbers $w \in C_{G_t} \setminus \{w_0\}$ belong to the variable spheres referred to $(Otxy)_t$ and centered in the complex point O.

q_4). For $x = 0$, we obtain the complex number $w = t + iy$, with the imagine in the complex plane (Oty), $i^2 = -1$.

q_5). For $y = 0$, we obtain the complex number $w = t + kx$, with image in the complex plane (Otx), $k^2 = -1$.

q_6). For $t = 0$, we have the imaginary generalized complex number $w = kx + iy$, $k^2 = i^2 = -1$, with the imagine in the imaginary complex plane (Oxy).

Definition 1.7. *The generalized complex number* $w_1, w_2 \in C_{G_x}$, $w_1 = x_1 + iy_1 + jt_1$, $w_2 = x_2 + iy_2 + jt_2$, $x_i, y_i, t_i \in R^3$, $i = \overline{1,2}$, *are equal,* $w_1 = w_2$, *if*

$$\begin{cases} x_1 & = & x_2 \\ y_1 & = & y_2 \\ t_1 & = & t_2 \end{cases} \tag{21}$$

The generalized complex numbers $w_1, w_2 \in C_{G_x}$, $w_1 = ai$, $w_2 = bj$, $a, b \in R^*$, $i = j = \sqrt{-1}$, *are equal,* $w_1 = w_2$, *if*

$$a = b. \tag{22}$$

Analogous, we define the equality of two generalized complex numbers in C_{G_y}, C_{G_t}.

The set of generalized complex numbers Pescar, with the notation C_{G_p} is defined by

$$C_{G_p} = C_{G_x} \bigcup C_{G_y} \bigcup C_{G_t}. \tag{23}$$

2. Addition of generalized complex numbers

Let be the set of generalized complex numbers C_{G_x}. We define the intern addition operation, with the notation "+" such that "+": $C_{G_x} \times C_{G_x} \to C_{G_x}$, by to rule $(w_1, w_2) \in C_{G_x} \times C_{G_x} \overset{"+"}{\to} w_1 + w_2 \in C_{G_x}$,

$$w_1 = x_1 + iy_1 + jt_1, \ w_2 = x_2 + iy_2 + jt_2,$$

then

$$w_1 + w_2 \overset{\text{definition}}{=} x_1 + x_2 + (y_1 + y_2)\, i + (t_1 + t_2)\, j. \tag{24}$$

$(C_{G_x}, +)$ is commutative group, because are satisfied the conditions:
1) $w_1 + w_2 = w_2 + w_1$, $\forall\, w_1, w_2 \in C_{G_x}$ (commutativity).

We have

$$x_1 + x_2 + i\,(y_1 + y_2) + j(t_1 + t_2) \;=\; x_2 + x_1 + i\,(y_2 + y_1) + j\,(t_2 + t_1)$$

and from (21) we obtain

$$\begin{cases} x_1 + x_2 &= x_2 + x_1 \\ y_1 + y_2 &= y_2 + y_1 \\ t_1 + t_2 &= t_2 + t_1. \end{cases} \tag{25}$$

Since $(R, +)$ is commutative group, the relations (25) are true.

2) $w_1 + (w_2 + w_3) = (w_1 + w_2) + w_3,\ \forall\, w_1, w_2, w_3 \in C_{G_x}$(associativity).
We obtain

$$\begin{aligned} x_1 \;&+\; (x_2 + x_3) = (x_1 + x_2) + x_3 \\ y_1 \;&+\; (y_2 + y_3) = (y_1 + y_2) + y_3 \\ t_1 \;&+\; (t_2 + t_3) = (t_1 + t_2) + t_3, \end{aligned} \tag{26}$$

true relations, because $(R, +)$ is group.

3) $\exists\ e \in C_{G_x},\ e = e_1 + e_2 i + e_3 j,\ (e_1, e_2, e_3) \in R^3$ ("e" neutral element), such that
$w + e = e + w = w,\ \forall w \in C_{G_x},\ w = x + iy + jt.$

Let $w + e = w \iff x + e_1 + i\,(y + e_2) + j\,(t + e_3) = x + iy + jt$ and using (21), we have

$$\begin{cases} x + e_1 &= x \\ y + e_2 &= y \\ t + e_3 &= t. \end{cases} \tag{27}$$

obtaining

$$e_1 = 0,\ e_2 = 0,\ e_3 = 0,\ e = 0 + i0 + j0 \in C_{G_x}.$$

The relation $w + e = e + w$ is true by the condition of commutativity 1).

Neutral element of the set C_{G_x} is "e" and we have $e = 0{+}i0{+}j0,\ (\,0, 0, 0\,) \in R^3$.

Neutral element $e \in C_{G_x}$, is the zero generalized complex number, $e = w_0 \in C_{G_x}$.

4) $\forall\ w \in C_{G_x},\ \exists w_s \in C_{G_x}$ such that $w + w_s = w_s + w = e$
("w_s" symmetrical element for the element $w \in C_{G_x}$).

We determine $w_s = x_s + iy_s + jt_s,\ (x_s,\ y_s,\ t_s) \in R^3$
and
$w + w_s\ =\ e \Leftrightarrow x + x_s + i\,(y + y_s)\ +\ j\,(t + t_s)\ =\ 0\ +\ i0\ +\ j0.$
We obtain

$$\begin{cases} x + x_s &= 0 \\ y + y_s &= 0 \\ t + t_s &= 0 \end{cases} \tag{28}$$

and hence
$x_s = -x,\ \ y_s = -y,\ \ t_s = -t,$
$w_s = -x + i\,(-y)\ +\ j\,(-t)\,,\ \ (-x,\ -y,\ -t)\ \in R^3,\ \ w_s \in C_{G_x}.$
The relation $w\ +\ w_s\ =\ w_s\ +\ w$ is true by the commutativity 1).

Definition 2.1. *The symmetrical generalized complex number w_s is called generalized complex number opposite for $w\ \in\ G_x,\ w_s\ =\ -w,\ w_s\ \in C_{G_x}$.*

3. Multiplication of generalized complex numbers

Definition 3.1. *The multiplication of real number $a\ \in\ R$ by $w\ \in\ C_{G_x},\ \ w\ =\ x\ +\ yi\ +\ tj$, with the notation "$\times$", is defined by:*

$$a\ \times\ w\ =\ a\ \times\ x\ +\ a\ \times\ yi\ +\ a\ \times\ tj. \tag{29}$$

Remark 3.2.
From the relations (6) we obtain

$$a\ \times\ w\ =\ a\ \cdot\ x\ +\ a\ \cdot\ yi\ +\ a\ \cdot\ tj. \tag{30}$$

Definition 3.3. *Let be generalized complex numbers Pescar, $w_1\ \in\ C_{G_x}$, $w_2\ \in\ C_{G_x},\ w_1 = x_1 + y_1 i + t_1 j,\ w_2 = x_2 + y_2 i + t_2 j$. The multiplication of complex numbers w_1, w_2 having notation "$\times$" is the generalized complex number $w\ \in\ C_{G_x}$,*

$$w = w_1 \times w_2 = x_1x_2 - y_1y_2 - t_1t_2 + (x_1y_2 + x_2y_1)\,i + (x_1t_2 + x_2t_1)\,j. \tag{31}$$

Using the relations (6), we have

$$w = w_1 \times w_2 = (x_1 + y_1 i + t_1 j) \times (x_2 + y_2 i + t_2 j) =$$

$$= x_1 \times x_2 + x_1 \times y_2 i + x_1 \times t_2 j + y_1 i \times x_2 + y_1 i \times y_2 i +$$

$$+ y_1 i \times t_2 j + t_1 j \times x_2 + t_1 j \times y_2 i + t_1 j \times t_2 j =$$

$$= x_1 x_2 - y_1 y_2 - t_1 t_2 + (x_1 y_2 + x_2 y_1)\, i + (x_1 t_2 + x_2 t_1)\, j.$$

Remark 3.4.
If $w_1 \in C_{G_y}$, $w_2 \in C_{G_y}$, $w_1 = y_1 + t_1 j + x_1 k$, $w_2 = y_2 + t_2 j + x_2 k$, then $w_1 \times w_2 = w \in C_{G_y}$ and we have

$$w = w_1 \times w_2 = y_1 y_2 - t_1 t_2 - x_1 x_2 + (y_1 t_2 + t_1 y_2)\, j + (x_1 y_2 + x_2 y_1)\, k. \quad (32)$$

Remark 3.5.
If $w_1 \in C_{G_t}$, $w_2 \in C_{G_t}$, $w_1 = t_1 + x_1 k + y_1 i$, $w_2 = t_2 + x_2 k + y_2 i$, then $w_1 \times w_2 = w \in C_{G_t}$ and we have

$$w = w_1 \times w_2 = t_1 t_2 - x_1 x_2 - y_1 y_2 + (x_1 t_2 + x_2 t_1)\, k + (y_1 t_2 + y_2 t_1)\, i. \quad (33)$$

4. The conjugated of generalized complex number Pescar

Definition 4.1. *Conjugated of generalized complex number $w \in C_{G_x}$, $w = x + yi + tj$, is the generalized complex number, with the notation $\overline{w} \in C_{G_x}$,*

$$\overline{w} = x - yi - tj. \quad (34)$$

5. The modulus of generalized complex number

Definition 5.1. *Let the complex point O and M the image of generalized complex number $w \in C_{G_x}$, $w = x + yi + tj$, $M(x, y, t)$ with respect to the Cartesian axes system $(Oxyt)_x$. The module of generalized complex number w, with the notation $|w|$, is distance of the point O, at the point M, $O(0,0,0)$ with notation* dist(O, M).

Remark 5.2.
Let $w \in C_{G_x}$, $w = x + yi + tj$. We have $|w| =$ dist (O, M) and from rectangular triangle OMM', figure 1.1, we obtain

$$|w| = \sqrt{x^2 + y^2 + t^2}. \quad (35)$$

Remark 5.3.

Let $w \in C_{G_y},\ w = y + tj + xk$, then from rectangular triangle OMM', figure 1.2, we obtain

$$|w| = \sqrt{y^2 + t^2 + x^2}. \tag{36}$$

Remark 5.4.

Let $w \in C_{G_t},\ w = t + xk + yi$, then from rectangular triangle OMM', figure 1.3, we obtain

$$|w| = \sqrt{t^2 + x^2 + y^2}. \tag{37}$$

Remark 5.5.

Let be complex numbers $w,\, \overline{w} \in C_{G_x}$, then we have

$$w \times \overline{w} = |w|^2. \tag{38}$$

Proof.

We have $w = x + yi + tj$, $\overline{w} = x - yi - tj$ and $w \times \overline{w} = x^2 + y^2 + t^2 = |w|^2$.

6. The properties of generalized complex nhumbers $(C_{G_x} \setminus \{0\}, \times)$

Let be the set of generalized complex numbers $C_{G_x} \setminus \{0\},\ 0 = 0 + 0i + 0j$ and the law of intern composition "$\times$", multiplication of generalized complex numbers such that "$\times$": $\{C_{G_x} \setminus \{0\}\} \times \{C_{G_x} \setminus \{0\}\} \to C_{G_x} \setminus \{0\}$, defined by rule:

$(w_1,\, w_2) \in \{C_{G_x} \setminus \{0\}\} \times \{C_{G_x} \setminus \{0\}\} \overset{"\times"}{\to} w_1 \times w_2 \in C_{G_x} \setminus \{0\}$,

$$w_1 = x_1 + y_1 i + t_1 j,\ \ w_2 = x_2 + y_2 i + t_2 j,$$

then

$$w_1 \times w_2 \overset{\text{definition 3.3}}{=} x_1 x_2 - y_1 y_2 - t_1 t_2 + (x_1 y_2 + x_2 y_1)\, i + (x_1 t_2 + x_2 t_1)\, j.$$

$(C_{G_x} \setminus \{0\}, \times)$ satisfies the properties:

1) $w_1 \times w_2 = w_2 \times w_1,\ \forall\, w_1, w_2 \in C_{G_x} \setminus \{0\}$ (commutativity).

We have

$$x_1 x_2 - y_1 y_2 - t_1 t_2 + (x_1 y_2 + x_2 y_1)\, i + (x_1 t_2 + x_2 t_1) j =$$

$$= x_2 x_1 - y_2 y_1 - t_2 t_1 + (y_2 x_1 + y_1 x_2)\, i + (t_2 x_1 + t_1 x_2)\, j.$$

From (21) we obtain

$$\begin{cases} x_1x_2 - y_1y_2 - t_1t_2 & = & x_2x_1 - y_2y_1 - t_2t_1 \\ x_1y_2 + x_2y_1 & = & y_2x_1 + y_1x_2 \\ x_1t_2 + x_2t_1 & = & t_2x_1 + t_1x_2. \end{cases} \tag{39}$$

Since $(R^*, \bullet)$ is commutative group and $(R^*, +)$ is commutative group the relations (39) are true.

2) $\exists\ e \in C_{G_x} \backslash \{0\}$ ("e" neutral element), such that $\forall w \in C_{G_x} \backslash \{0\}$, we have: $w \times e = e \times w = w$.

We determine "e" of the form $e = a + bi + cj$. Let $w \in C_{G_x} \backslash \{0\}$, $w = x + yi + tj$ and $w \times e = w$.

We obtain: $ax - by - ct + (ay + bx)\, i + (at + cx)\, j = x + yi + tj$ and hence

$$\begin{cases} ax - by - ct & = & x \\ bx + ay & = & y \\ cx + at & = & t. \end{cases} \tag{40}$$

We obtain

$$a = 1,\ b = 0,\ c = 0 and e = 1 + 0i + 0j \in C_{G_x} \backslash \{0\}.$$

The relation $w \times e = e \times w$ is true by the condition of commutativity in $(C_{G_x} \backslash \{0\}, \times)$.

The generalized complex number $e = 1 + 0i + 0j$ is neutral element in $(C_{G_x} \backslash \{0\}, \times)$.

3) $\forall\ w \in C_{G_x} \backslash \{0\}$, $\exists w' \in C_{G_x} \backslash \{0\}$ (w' symmetrical element), such that $w \times w' = w' \times w = e$.

Let $w = x + yi + tj \in C_{G_x} \backslash \{0\}$ and we determine $w' = p + qi + rj$, $w' \in C_{G_x} \backslash \{0\}$.

We consider $w \times w' = e$ and we obtain $xp - yq - tr + (xq + yp)\, i + (xr + tp)\, j = 1 + 0i + 0j$, and hence we have

$$\begin{cases} xp - yq - tr & = & 1 \\ xq + yp & = & 0 \\ xr + tp & = & 0. \end{cases} \tag{41}$$

We obtain

$$\begin{cases} p &= \frac{x}{x^2+y^2+t^2} \\ q &= -\frac{y}{x^2+y^2+t^2} \\ r &= -\frac{t}{x^2+y^2+t^2}, \end{cases} \tag{42}$$

$x^2+y^2+t^2 \neq 0$, since $w \in C_{G_x} \backslash \{0\}$.
The symmetrical element for $w \in C_{G_x} \backslash \{0\}$ is

$$w' = \frac{x}{x^2+y^2+t^2} - \frac{y}{x^2+y^2+t^2} i - \frac{t}{x^2+y^2+t^2} j \in C_{G_x} \backslash \{0\}.$$

Relation $w \times w' = w' \times w$ is true, since commutativity in $(C_{G_x} \backslash \{0\}, \times)$.

The generalized complex number Pescar,

$$w' = \frac{x - yi - tj}{x^2+y^2+t^2}$$

is the symmetrical element for $w \in C_{G_x} \backslash \{0\},\ w = x + yi + tj$.

Remark 6.1.
We observe that

$$w' = \frac{\overline{w}}{|w|^2}.$$

7. The division of generalized complex numbers

Definition 7.1. *Let the generalized complex numbers Pescar,* $w_1 \in C_{G_x}$, $w_2 \in C_{G_x} \backslash \{0\}$,

$$w_1 = x_1 + y_1 i + t_1 j,\ \ w_2 = x_2 + y_2 i + t_2 j.$$

The division of generalized complex numbers w_1, w_2 *is defined by*

$$\frac{w_1}{w_2} = \frac{w_1 \times \overline{w_2}}{w_2 \times \overline{w_2}} = \frac{w_1 \times \overline{w_2}}{|\ w_2\ |^2}. \tag{43}$$

From (43) we obtain

$$\frac{w_1}{w_2} = \frac{w_1 \times \overline{w_2}}{\mid w_2 \mid^2} = \frac{x_1x_2 + y_1y_2 + t_1t_2}{x_2^2 + y_2^2 + t_2^2} + \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2 + t_2^2}i + \frac{t_1x_2 - x_1t_2}{x_2^2 + y_2^2 + t_2^2}j. \quad (44)$$

Remark 7.2.
Analogous we define the division of generalized complex numbers in set of C_{G_y} and C_{G_t}.

8. Raising to power of generalized complex number Pescar

Definition 8.1. *Let be* $w \in C_{G_x},\ n \in N^*$. *By definition*

$$w^n = \underbrace{w \times w \times \ldots w}_{n\text{times}} \in C_{G_x}.$$

Remark 8.2.

Analogous we define the raising to power of generalized complex numbers from C_{G_y} and C_{G_t}.

9. The trigonometric form of generalized complex numbers

Let be the set of generalized complex numbers C_{G_x} and $w \in C_{G_x},\ w = x + yi + tj$.
From the figure 1.1 we obtain

$$\begin{cases} x &= \rho\ cosu \\ y &= \rho\ sinu \\ \rho &= |w| sinv \\ t &= |w| cosv. \end{cases} \quad (45)$$

We have $M \in (\Sigma_r)$ and $|w| = r$, where r is radius of sphere (Σ_r) , then we obtain

$$\begin{cases} x &= r\ cosu\ sinv \\ y &= r\ sinu\ sinv,\ u \in [0,\ 2\pi\ ],\ v\ \in\ [\ 0, \pi\ ] \\ t &= r\ cosv. \end{cases} \quad (46)$$

$u = \angle(Ox,\ OM')\ \ v = \angle(Ot,\ OM)$.

For the generalized complex number $w \in C_{G_x}$ we obtain the trigonometric form

$$w = r\ \left(cosu\ sinv + i\ sinu\ sinv + j\ cosv\right),\ |w| = r. \tag{47}$$

Remark 9.1.

If $u = 0$, then we obtain the complex number from C with the trigonometric form

$$w = z = r\ \left[\ cos\left(\frac{\pi}{2} - v\right)\ +\ j\ sin\left(\frac{\pi}{2} - v\right)\right],\ v\ \in\ \left[0,\ \frac{\pi}{2}\right]. \tag{48}$$

Remark 9.2.

If $v = \frac{\pi}{2}$, then we obtain the complex number from C with the trigonometric form and the image in complex plane (Oxy).

$$w = z = r\ \left(cosu + i\,sinu\right),\ u\ \in\ [0, 2\pi]. \tag{49}$$

Remark 9.3.

Analogous we define the trigonometric form of complex numbers from C_{G_y} and C_{G_t}.

In C_{G_y}, for $w\ \in\ C_{G_y},\ w = y + tj + xk$ we have

$$w = r\ \left(cosu\,sinv + j\,sinu\,sinv + k\,cosv\right),\ u\ \in\ [0, 2\pi],\ v\ \in\ \left[0,\ \frac{\pi}{2}\right]. \tag{50}$$

In C_{G_t}, for $w\ \in\ C_{G_t},\ w = t + xk + yi$ we have

$$w = r\ \left(cosu\,sinv + k\,sinu\,sinv + i\,cosv\right),\ u\ \in\ [o, 2\pi],\ v\ \in\ \left[0,\ \frac{\pi}{2}\right]. \tag{51}$$

10. Properties of modulus generalized complex numbers

p_1) Let be the generalized complex numbers $w_1\ \in\ C_{G_x}$, $w_2\ \in\ C_{G_x}$,

$$w_1 = x_1 + y_1 i + t_1 j,\ \ w_2 = x_2 + y_2 i + t_2 j.$$

We have

$$|w_1 + w_2|\ \leq\ |w_1| + |w_2|. \tag{52}$$

Proof. The relation (52) is equivalent with

$$\sqrt{(x_1+x_2)^2+(y_1+y_2)^2+(t_1+t_2)^2} \le \sqrt{x_1^2+y_1^2+t_1^2}+\sqrt{x_2^2+y_2^2+t_2^2}$$

or

$$x_1x_2+y_1y_2+t_1t_2 \le \sqrt{x_1^2+y_1^2+t_1^2} \cdot \sqrt{x_2^2+y_2^2+t_2^2}. \qquad (53)$$

Case 1. If $x_1x_2+y_1y_2+t_1t_2 \le 0$ the relation (53) is true.
Case 2. If $x_1x_2+y_1y_2+t_1t_2 \ge 0$ from the relation (53) we obtain

$$2\left(x_1x_2y_1y_2+x_1x_2t_1t_2+y_1y_2t_1t_2\right) \le$$
$$\le (x_1y_2)^2+(y_1x_2)^2+(x_1t_2)^2+(t_1x_2)^2+(y_1t_2)^2+(y_2t_1)^2$$

and hence, we have

$$(x_1y_2-y_1x_2)^2+(x_1t_2-t_1x_2)^2+(y_1t_2-y_2t_1)^2 \ge 0.$$

So, the relation (52) is true.

p_2) Let be $w_1 \in C_{G_x}$, $w_2 \in C_{G_x}$, $w_1=x_1+y_1i+t_1j$, $w_2=x_2+y_2i+t_2j$. We have

$$|w_1 \times w_2| \le |w_1| \cdot |w_2|. \qquad (54)$$

Proof. The relation (54) is equivalent with

$$2y_1y_2t_1t_2 \le (y_1t_2)^2+(t_1y_2)^2$$

or

$$(y_1t_2-t_1y_2)^2 \ge 0$$

and the property p_2) is true.

p_3) Let be $w_1 \in C_{G_x}$, $w_2 \in C_{G_x}\backslash\{0\}$, $w_1=x_1+y_1i+t_1j$, $w_2=x_2+y_2i+t_2j$. We have

$$\left|\frac{w_1}{w_2}\right| \le \frac{|w_1|}{|w_2|}. \qquad (55)$$

Proof.

We have

$$\left|\frac{w_1}{w_2}\right| = \frac{|w_1 \times \overline{w_2}|}{|\ w_2\ |^2} \le \frac{|w_1| \cdot |\overline{w}_2|}{|\ w_2\ |^2} = \frac{|w_1|}{|w_2|}.$$

p_4) Let be $w_1 \in C_{G_x}$, $n \in N^*$. We have

$$|w^n| \le |w|^n. \tag{56}$$

Proof. We have

$$|w^n| = |w \times w \times \ldots w| \le \underbrace{|w| \cdot |w| \cdot \cdots \cdot |w|}_{n\text{times}} = |w|^n.$$

11. General complex numbers

Let be $C_{G_p} = C_{G_x} \bigcup C_{G_y} \bigcup C_{G_t}$ and $w_1 \in C_{G_x}$, $w_2 \in C_{G_y}$, $w_3 \in C_{G_t}$,

$$w_1 = x_1 + y_1 i + t_1 j,\ \ w_2 = y_2 + t_2 j + x_2 k,\ \ w_3 = t_3 + x_3 k + y_3 i.$$

We define addition of generalized complex numbers $w_1,\ w_2,\ w_3$ thus

$$w_1 + w_2 + w_3 \overset{\text{def.}}{=} x_1 + y_2 + t_3 + (y_1 + y_3)\ i + (t_1 + t_2)\ j + (x_2 + x_3)\ k. \tag{57}$$

Therefore

$$w_1 + w_2 + w_3 = a + bi + cj + dk,\ a, b, c, d \in R.$$

Definition 11.1 *We usually call the set of general complex numbers, the set:*

$$VP = \left\{w = a + bi + cj + dk,\ a, b, c, d \in R,\ i^2 = j^2 = k^2 = -1\right\}. \tag{58}$$

Definition 11.2 *Let be* $w_1 \in C_{G_x}$, $w_2 \in C_{G_y}$, $w_3 \in C_{G_t}$

$$w_1 = x_1 + y_1 i + t_1 j,\ \ w_2 = y_2 + t_2 j + x_2 k,\ \ w_3 = t_3 + x_3 k + y_3 i.$$

The multiplication of generalized complex numbers w_1, w_2, w_3 *is the ge-neral complex number*

$$w \in VP,\ w = w_1 \times w_2 \times w_3 = p + qi + rj + sk,\ p, q, r, s \in R. \tag{59}$$

Definition 11.3 *The modulus of general complex number* $w \in VP$, $w = a + bi + cj + dk$ *is defined by*

$$|w| = \sqrt{a^2 + b^2 + c^2 + d^2}. \tag{60}$$

Definition 11.4 *The conjugate of general complex number* $w \in VP$, $w = a + bi + cj + dk$, *is the general complex number* $\overline{w} \in VP$, $\overline{w} = a - bi - cj - dk$.

Remark 11.5.

Let be $w, \overline{w} \in VP$, then

$$w \times \overline{w} = |w|^2. \tag{61}$$

Definition 11.6 *Let be* $u \in VP$, $v \in VP \setminus \{0\}$, $u = a_1 + b_1 i + c_1 j + d_1 k$, $v = a_2 + b_2 i + c_2 j + d_2 k$. *The division of general complex numbers* $\frac{u}{v}$ *is defined by*

$$\frac{u}{v} \overset{def.}{=} \frac{u \times \overline{v}}{v \times \overline{v}} = \frac{u \times \overline{v}}{|v|^2}. \tag{62}$$

References

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