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AN INVERSE THEOREM IN SIMULTANEOUS APPROXIMATION FOR A LINEAR COMBINATION OF BERNSTEIN-DURRMEYER TYPE POLYNOMIALS

Karunesh Singh and P.N. Agrawal

Abstract. The present paper is a continuation of our work in [5]. Here we study an inverse result in simultaneous approximation for a linear combination of Bernstein-Durrmeyer type polynomials by Peetre's K- functional approach.

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1. Introduction

The Bernstein-Durrmeyer type polynomial operators

$$P_n(f;x) = n\sum_{\nu=1}^{n} p_{n,\nu}(x)\int_0^1 p_{n-1,\nu-1}(t)f(t)\,du + (1-x)^n f(0),$$

where

$$p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu},\ \ 0 \leqslant x \leqslant 1,$$

defined on $L_B[0,1]$, the space of bounded and Lebesgue integrable functions on $[0,1]$ were introduced by Gupta and Maheshwari [3] wherein they studied the approximation of functions of bounded variation by these operators. In [1] Gupta and Ispir studied the pointwise convergence and Voronovskaja-type asymptotic results in simultaneous approximation for these operators.
For $f \in L_B[0,1]$, the operators $P_n(f;x)$ can be expressed as

$$P_n(f;x) = \int_0^1 W_n(t,x)f(t)\,dt,$$

where

$$W_n(t,x) = n\sum_{\nu=1}^{n} p_{n,\nu}(x)\, p_{n-1,\nu-1}(t) + (1-x)^n\delta(t),$$

$\delta(t)$ being the Dirac-delta function, is the kernel of the operators.
It turns out that the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be. In order to speed up the rate of convergence by the operators P_n, we considered the linear combination $P_n(f,k,.)$ of the operators P_n, as

$$P_n(f,k,x) = \sum_{j=0}^{k} C(j,k)P_{d_jn}(f,x),$$

where

$$C(j,k) = \prod_{i=0, i\neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0,0) = 1,$$

$d_0, d_1, ...d_k$ being $(k+1)$ arbitrary but fixed distinct positive integers.

Throughout this paper, we assume $C(I)$ the space of all continuous functions on the interval I, $\|.\|_{C(I)}$ the sup norm on the space $C(I)$ and C a constant not necessarily the same in the different cases.
Let $I = [a,b]$ be a fixed subinterval of $(0,1)$, $I' = [a',b'] \subset (a,b)$ and $I'' = [a'',b''] \subset (a',b')$. Further, let $G^r(I') = \{g \in C_0^r : supp\ g \ \subset I'\}$.

For $f \in G^r(I')$ and $g \in G^{2k+r+2}(I')$ we define

$$K_r(\xi,f,I') = \inf_{g\in G^{2k+r+2}(I')}\left\{\|f^{(r)} - g^{(r)}\|_{C(I')} + \xi\left(\|g^{(r)}\|_{C(I')} + \|g^{(2k+2+r)}\|_{C(I')}\right)\right\},$$

where $0 < \xi \leq 1$.

For $0 < \beta < 2$, we define $C_0^r(\beta, k+1, I')$ as the class of all $f \in G^r(I')$ such that the functional

$$\|f\|_{\beta,r} = \sup_{0<\xi\leq 1} \xi^{-\beta/2}K_r(\xi,f,I') < C, \text{ for some } C > 0.$$

2. Auxiliary Results

In this section we give some results which are useful in establishing our main theorem.

For sufficiently small $\eta > 0, 0 < a_1 < a_2 < b_2 < b_1 < 1, I_i = [a_i, b_i], i = 1, 2$ and $m \in \mathbb{N}$, the Steklov mean $f_{\eta,m}$ of $m-$th order corresponding to $f \in C[a, b]$ is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left(f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m t_i} f(t)\right) \prod_{i=1}^m dt_i, \ , t \in I_1,$$

where Δ^m_h is the $m-$th order forward difference operator with step length h.

Lemma 1. *[4] For the function $f_{\eta,m}$, we have*

(a) $f_{\eta,m}$ has derivatives up to order m over I_1;

(b) $\|f^{(r)}_{\eta,m}\|_{C(I_1)} \leqslant C_r \ \omega_r(f, \eta, [a, b]), r = 1, 2, ..., m$;

(c) $\|f - f_{\eta,m}\|_{C(I_1)} \leqslant C_{m+1} \ \omega_m(f, \eta, [a, b])$;

(d) $\|f_{\eta,m}\|_{C(I_1)} \leqslant C_{m+2} \ \eta^{-m}\|f\|_{C[a,b]}$;

(e) $\|f^{(r)}_{\eta,m}\|_{C(I_1)} \leqslant C_{m+3} \ \|f\|_{C[a,b]}$,
where $C_i's$ are certain constants that depend on i but are independent of f and η.

Lemma 2. *[4] For the function $p_{n,\nu}(t)$, there holds the result*

$$t^r(1-t)^r \frac{d^r}{dt^r}(p_{n,\nu}(t)) = \sum_{\substack{2i+j \leqslant r \\ i,j \geq 0}} n^i(\nu - nt)^j q_{i,j,r}(t) p_{n,\nu}(t),$$

where $q_{i,j,r}(t)$ are certain polynomials in t independent of n and ν.

Lemma 3. *[1] For the function $u_{n,m}(t), m \in \mathbb{N}^0$ (the set of non-negative integers) defined as*

$$u_{n,m}(t) = \sum_{\nu=1}^{n} p_{n,\nu}(t) \left(\frac{\nu}{n} - t\right)^m,$$

we have $u_{n,0}(t) = 1$ and $u_{n,1}(t) = 0$. Further, there holds the recurrence relation

$$nu_{n,m+1}(t) = t\left[u'_{n,m}(t) + mu_{n,m-1}(t)\right], m = 1, 2, 3, ...$$

Consequently,
(i) $u_{n,m}(t)$ is a polynomial in t of degree at most m;
(ii) for every $t \in [0, \infty), u_{n,m}(t) = O\left(n^{-[(m+1)/2]}\right)$, where $[\alpha]$ denotes the integral part of α.

Lemma 4. *[4] For the function $\mu_{n,m}(t)$, we have $\mu_{n,0}(t) = 1$, $\mu_{n,1}(t) = \frac{(-t)}{(n+1)}$ and there holds the recurrence relation*

$$(n+m+1)\mu_{n,m+1}(t) = t(1-t)\left\{\mu'_{n,m}(t) + 2m\mu_{n,m-1}(t)\right\} + (m(1-2t)-t)\mu_{n,m}(t),$$

for $m \geq 1$.
Consequently, we have
(i) $\mu_{n,m}(t)$ is a polynomial in t of degree m;
(ii) for every $t \in [0,1], \mu_{n,m}(t) = \mathcal{O}\left(n^{-[(m+1)/2]}\right)$, where $[\beta]$ is the integer part of β.

Theorem 1. *[5] Let $f \in L_B[0,1]$ admitting a derivative of order $(2k+r+2)$ at a point $x \in (0,1)$ then we have*

$$\lim_{n\to\infty} n^{k+1}[P_n^{(r)}(f,k,x) - f^{(r)}(x)] = \sum_{\nu=r}^{2k+r+2} \frac{f^{(\nu)}(x)}{\nu!} Q(\nu,k,r,x) \tag{1}$$

and

$$\lim_{n\to\infty} n^{k+1}[P_n^{(r)}(f,k,x) - f^{(r)}(x)] = 0, \tag{2}$$

where $Q(\nu,k,r,x)$ are certain polynomials in x of degree ν. Further, the limits in (1) and (2) hold uniformly in $[a,b]$ if $f^{(2k+r+2)}$ is continuous on $(a-\eta, b+\eta) \subset (0,1)$, $\eta > 0$.

Lemma 5. *If $f^{(r)} \in G^r(I'')$ and*

$$\left\|P_n^{(r)}(f,k,.) - f^{(r)}\right\| \leq Cn^{-\beta(k+1)/2},$$

then

$$K_r(\xi,f,I') \leq C\left(n^{-\beta(k+1)/2} + n^{k+1}\xi K_r(n^{-(k+1)},f,I')\right). \tag{3}$$

Consequenlty, $K_r(\xi,f,I') \leq C\xi^{\beta/2}$ i.e. $f \in C_0^r(\beta,k+1,I')$

Proof. Following [6] it is enough to show that 3 holds for all n sufficiently large. Since supp$f \subset I''$, in view of Theorem 1 [5], we can find a function $h \in G^{2k+r+2}(I')$ such that for $i = r$ and $2k+2+r$, there holds

$$\|h^{(i)} - P_n^{(i)}(f,k,.)\|_{C(I)} \leq Cn^{-(k+1)}, \text{ for all } n \text{ sufficiently large.}$$

Therefore,

$$\begin{aligned} K_r(\xi,f,I') &\leq 3Cn^{-(k+1)} + \|f^{(r)} - P_n^{(i)}(f,k,.)\|_{C(I)} \\ &+ \xi\left(\|P_n^{(r)}(f,k,.)\| + \|P_n^{(2k+r+2r)}(f,k,.)\|_{C(I)}\right) \end{aligned}$$

Hence, it is sufficient to show that for each $g \in G^{2k+r+2}(I'))$,

$$\|P_n^{(2k+2+r)}(f,k,.)\|_{C(I')} \leq Cn^{(k+1)}\left(\|f^{(r)}-g^{(r)}\|_{C(I)}+n^{-(k+1)}g^{(2k+2+r)}\right). \quad (4)$$

Now,

$$\begin{aligned}\|P_n^{(2k+2+r)}(f,k,.)\|_{C(I')} &\leq \|P_n^{(2k+2+r)}(f-g,k,.)\|_{C(I')}+\|P_n^{(2k+2+r)}(g,k,.)\|_{C(I')}\\ &= \Sigma_1+\Sigma_2.\end{aligned}$$

By using Taylor's expansion of $(f-g)(t)$ about $t=x$, Lemmas 4, 2, Schwarz inequality for integration and then for summation, we get

$$\Sigma_1 \leq Cn^{(k+1)}\|f^{(r)}-g^{(r)}\|_{C(I)} \quad (\text{supp}f \cup \text{supp}g \subset I')$$

Similarly, using the Taylor's expansion of $g(t)$ about $t=x$, we get

$$\Sigma_2 \leq C\|g^{(2k+2+r)}\|_{C(I)}.$$

Combining the estimates of Σ_1 and Σ_2, the inequality 4 follows. Hence, 3 holds.

□

Lemma 6. *If $f^{(r)} \in G^r(I'')$ and $f \in C_0^r(\beta,k+1,I')$ then for sufficiently large n, we have*

$$\left\|P_n^{(r)}(f,k,.)-f^{(r)}\right\|_{C(I)} = O(n^{-\beta(k+1)/2}).$$

Proof. For $g \in G^{2k+r+2}(I')$, we have

$$\begin{aligned}\left\|P_n^{(r)}(f,k,.)-f^{(r)}\right\|_{C(I)} &\leq \left\|P_n^{(r)}(f-g,k,.)\right\|_{C(I)}+\left\|P_n^{(r)}(g,k,.)-f^{(r)}\right\|_{C(I)}\\ &= \Sigma_1+\Sigma_2.\end{aligned}$$

Proceeding along the lines of estimate Σ_1 in Lemma 6 and in view of $\text{supp}(f-g) \subset I'$, we get

$$\Sigma_1 \leq C\|f^{(r)}-g^{(r)}\|_{C(I)}.$$

Using Theorem 1 and intermediate derivative property ([2], p.5) we obtain

$$\Sigma_2 \leq C\|f^{(r)}-g^{(r)}\|_{C(I)}+Cn^{-(k+1)}\left(\|g^{(r)}\|_{C(I')}+\|g^{(2k+2+r)}\|_{C(I')}\right).$$

Combining the estimates of Σ_1 and Σ_2, we get

$$\begin{aligned}\left\|P_n^{(r)}(f,k,.)-f^{(r)}\right\|_{C(I)} &\leq C.K_r(n^{-(k+1)},f;I') \\ &\leq C.O(n^{-\beta(k+1)/2}),\end{aligned}$$

since $f\in C_0^r(\beta,k+1,I')$.

□

Lemma 7. *If $f^{(r)}\in G^r(I'')$ then*

$$f\in C_0^r(\beta,k+1,I')\Leftrightarrow f^{(r)}\in Liz(\beta,k+1,I').$$

Proof. Let $|\delta|<h$ and $g\in G^{2k+r+2}(I')$. Then, if $f\in C_0^r(\beta,k+1,I')$ we get

$$\begin{aligned}|\Delta_\delta^{(2k+2)}f^{(r)}(x)| &\leq |\Delta_\delta^{(2k+2)}(f^{(r)}(x)-g^{(r)}(x))|+|\Delta_\delta^{(2k+2)}g^{(r)}(x)| \\ &\leq 2^{2k+2}\|f^{(r)}-g^{(r)}\|_{C(I')}+\delta^{(2k+2)}\|g^{(2k+2+r)}\|_{C(I')} \\ &\leq 2^{2k+2}C.K_r(\delta^{(2k+2)},f;I') \\ &\leq 2^{2k+2}C.\delta^{\beta(k+1)}.\end{aligned}$$

It follows that

$$\begin{aligned}\omega_{2k+2}(f^{(r)},h;I) &\leq \sup_{|\delta|\leq h}|\Delta_\delta^{(2k+2)}f^{(r)}(x)| \\ &\leq C.h^{\beta(k+1)}.\end{aligned}$$

i.e. $f^{(r)}\in \mathrm{Liz}\,(\beta,k+1,I')$.

Conversely, suppose that $f^{(r)}\in \mathrm{Liz}\,(\beta,k+1,I')$ and $f_{\eta,2k+r+2}$ be the $(2k+r+2)$th order Steklov mean corresponding to f as defined in . Hence $f_{\eta,2k+r+2}(x)\in G^{2k+r+2}\,(I')$, by property (b) of Lemma 1 we have

$$\begin{aligned}\|f_{\eta,2k+r+2}^{(2k+2+r)}(x)\|_{C(I')} &\leq C\eta^{-(2k+r+2)}\omega_{2k+r+2}(f,\eta;I) \\ &\leq C\eta^{-(2k+r+2)}\eta^r\omega_{2k+2}(f^{(r)},\eta;I) \\ &\leq C\eta^{-(2k+2)+\beta(k+1)}.\end{aligned}$$

Using property (c) of Lemma 1, we get

$$\|f_{\eta,2k+r+2}^{(r)}(x)-f^{(r)}\|_{C(I')}\leq C\omega_{2k+2}(f^{(r)},\eta;I))\leq Cn^{\beta(k+1)},$$

which implies that $f\in C_0^r(\beta,k+1,I')$. □

Theorem 2. *[5] Let $p \in \mathbb{N}, 1 \leq p \leq 2k+2$ and $f \in L_B[0,1]$. If $f^{(p+r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset [0,1]$, $\eta > 0$ then*

$$\left\|P_n^{(r)}(f,k,.) - f^{(r)}\right\|_{C(I)} \leq \max\left\{C_1 n^{-p/2}\omega\left(f^{(p+r)}, n^{-1/2}\right), C_2 n^{-(k+1)}\right\},$$

where $C_1 = C_1(k,p,r)$, $C_2 = C_2(k,p,r,f)$ and $\omega\left(f^{(p+r)}, \delta\right)$ is the modulus of continuity of $f^{(p+r)}$ on $(a-\eta,\, b+\eta)$.

3. Main Result

Theorem 3. *Let $f \in L_B[0,1]$ and $0 < \alpha < 2$. Then, in the following statements, the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) hold:*

(i) $\| P_n^{(r)}(f,k,.) - f^{(r)}\|_{C[a_1,b_1]} = O(n^{-\alpha(k+1)/2})$;

(ii) $f \in \mathrm{Liz}(\alpha, k, a_2, b_2)$;

(iii) (a) *for $m < \alpha(k+1) < m+1$, $m = 0,1,....2k-1$, $f^{(m)}$ exists and belongs to the class* $\mathrm{Lip}(\alpha k - m, a_2, b_2)$,
(b) *for $\alpha(k+1) = m+1$, $m = 0,1,....2k-2$, $f^{(m)}$ exists and belongs to the class* $\mathrm{Lip}^*(1, a_2, b_2)$;

(iv) $\| P_n^{(r)}(f,.) - f(.)\|_{C[a_3,b_3]} = O(n^{-\alpha(k+1)/2})$.

Proof. To show $(i) \Rightarrow (ii)$ we reassume that $a_1 < a' < a'' < a_2$, $b_2 < b' < b'' < b' < b_1$, $I = [a', b']$ and $I'' = [a'', b'']$. Writing $\tau = \beta(k+1)$, we first consider the case $0 < \tau \leq 1$.
Let $g \in C_0^\infty$ be such that supp$g \subset I''$ and $g(x) = 1$ on I_2. Writing $D \equiv \frac{d}{dx}$, then for $x \in I'$ we have

$$\begin{aligned}
P_n^{(r)}(fg,k,x) - (fg)^{(r)}(x) &= D^r\{P_n((fg)(t) - (fg)(x), k, x)\} \\
&= D^r\{P_n((f(t)(g(t) - g(x)), k, x)\} \\
&+ D^r\{P_n((g(x)(f(t) - f(x)), k, x)\} \\
&= \Gamma_1 + \Gamma_2, \text{ say.}
\end{aligned}$$

Using Leibnitz theorem

$$\begin{aligned}\Gamma_1 &= \sum_{j=0}^{k} C(j,k) D^r \left\{ \int_0^1 W_{d_j n}(t,x) f(t)(g(t)-g(x))dt \right\} \\ &= \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{r} \binom{r}{i} \int_0^1 W_{d_j n}^{(i)}(t,x) D^{r-i}\{f(t)(g(t)-g(x))\}dt \\ &= -\sum_{i=0}^{r-1} \binom{r}{i} g^{(r-i)}(x) P_n^{(i)}(f,k,x) \\ &+ \sum_{j=0}^{k} C(j,k) \int_0^1 W_{d_j n}^{(r)}(t,x) f(t)(g(t)-g(x))dt \\ &= J_1 + J_2, \text{ say.}\end{aligned}$$

By Theorem 2, we have

$$J_1 = -\sum_{i=0}^{r-1} \binom{r}{i} g^{(r-i)}(x) f^{(i)}(x) + O(n^{-\tau/2}), \text{ uniformly on } I'.$$

Next, we estimate J_2. By Taylor's expansion of f and g at $t = x$, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + o(t-x)^r$$

and

$$g(t) = \sum_{i=0}^{r+1} \frac{g^{(i)}(x)}{i!}(t-x)^i + o(t-x)^{r+1}.$$

Hence, by using Schwarz inequality and Lemma 4 we obtain

$$\begin{aligned}J_2 &= \sum_{i=1}^{r} \frac{f^{(r-i)}(x) g^{(i)}(x)}{i!(r-i)!} r! + O(n^{-1/2}) \\ &= \sum_{i=1}^{r} g^{(i)}(x) f^{(r-i)}(x) + O(n^{-\tau/2}),\end{aligned}$$

uniformly on I'.
Hence, $\Gamma_1 = O(n^{-\tau/2})$,uniformly on I'.

Again, by Leibnitz theorem,Theorem 2 and hypothesis (i) we obtain

$$\begin{aligned}\Gamma_2 &= \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{r} \binom{r}{i} \int_0^1 W_{d_j n}^{(i)}(t,x) D^{r-i}\{g(x)(f(t)-f(x))\}dt \\ &= \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)}(x) P_n^{(i)}(f,k,x) - (fg)^{(r)}(x) \\ &= O(n^{-\tau/2}), \text{ uniformly on } I'.\end{aligned}$$

Combining the estimates of Γ_1 and Γ_2 we obtain

$$\|P_n^{(r)}(fg,k,.) - (fg)^{(r)}\|_{C(I')} = O(n^{-\tau/2}).$$

Thus, by Lemma 5 and 7 we have

$$(fg)^{(r)} \in Liz(\beta, k+1, I'').$$

Hence, $f^{(r)} \in Liz(\beta, k+1, I_2)$ (in view of $g(x) = 1$ on I_2).
This completes the proof of the implication $(i) \Rightarrow (ii)$ when $0 < \tau \leq 1$.

Now to prove the implication $(i) \Rightarrow (ii)$ for $0 < \tau < 2k+2$, it is sufficient to assume it for $\tau \in (m-1, m)$ and prove it for $\tau \in [m, m+1)$, $m = 1, 2, ..., 2k+1$. Since, the result holds for $\tau \in (m-1, m)$, therefore $f^{(m+r-1)}$ exists and belongs to $Lip(1-\delta; [z_1, w_1])$ for any interval $[z_1, w_1] \subset (a_1, b_1)$ and $\delta > 0$.
Let z_2, w_2 be such that $I_2 \subset (z_2, w_2)$ and $[z_2, w_2] \subset (z_1, w_1)$. Let $g \in C_0^\infty$ be such that $g(x) = 1$ on I_2 and supp $g \in (z_2, w_2)$. Then, we have

$$\begin{aligned}\|P_n^{(r)}(fg,k,.) - (fg)^{(r)}\|_{C[z_2,w_2]} &\leq \|D^r\{P_n(g(x)(f(t)-f(x)),k,.\}\|_{C[z_2,w_2]} \\ &+ \|D^r\{P_n(f(t)(g(t)-g(x)),k,.\}\|_{C[z_2,w_2]} \\ &= \Sigma_3 + \Sigma_4, \text{ say.}\end{aligned}$$

Now, by Leibnitz theorem, Theorem 2 and assumption that (i) holds, we have

$$\begin{aligned}\Sigma_3 &= \|D^r\{g(x)P_n(f(t),k,.\} - (fg)^{(r)}\|_{C[z_2,w_2]} \\ &= \left\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)} P_n^{(i)}(f,k,.) - (fg)^{(r)} \right\|_{C[z_2,w_2]} \\ &= \left\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)} f^{(i)} - (fg)^{(r)} \right\|_{C[z_2,w_2]} + O(n^{-\tau/2}) \\ &= O(n^{-\tau/2}).\end{aligned}$$

Again, using Leibnitz theorem and Theorem 1, we obtain

$$\begin{aligned}\Sigma_4 &= \left\| -\sum_{i=0}^{r-1}\binom{r}{i} g^{(r-i)}(x)P_n^{(i)}(f(t),k,.) + P_n^{(r)}\left(f(t)(g(t)-g(x))\chi_2(t),k,.\right)\right\|_{C[z_2,w_2]} \\ &+ \quad o(n^{-(k+1)}) \\ &= \|J_3+J_4\|_{C[z_2,w_2]} + o(n^{-(k+1)}), \text{ say,}\end{aligned}$$

where $\chi_2(t)$ is the characteristic function of the interval $[z_1, w_1]$.
Then, by Theorem 2, we get

$$J_3 = -\sum_{i=0}^{r-1}\binom{r}{i} g^{(r-i)}(x)f^{(i)}(x) + O\left(n^{-(k+1)}\right),$$

uniformly on $[z_2, w_2]$.
Since, by the induction hypothesis $f^{(m+r-1)}$ exists and belongs to $\text{Lip}(1-\delta;[z_1,w_1])$ for any $\delta > 0$, by Taylor's expansion of f about $t = x$, we obtain

$$\begin{aligned}J_4 &= \sum_{j=0}^{k} C(j,k)\sum_{i=0}^{m+r-1}\frac{f^{(i)}(x)}{i!}\int_0^1 W_{d_jn}^{(r)}(t,x)(t-x)^i(g(t)-g(x))\chi_2(t)\,dt \\ &+ \sum_{j=0}^{k} C(j,k)\int_0^1 W_{d_jn}^{(r)}(t,x)\left(\frac{f^{(m+r-1)}(\xi)-f^{(m+r-1)}(x)}{(m+r-1)!}\right)\times \\ &\quad (t-x)^{m+r-1}(g(t)-g(x))\chi_2(t)dt \\ &= J_5 + J_6, \text{ say.}\end{aligned}$$

Using Theorem 1, we have

$$\begin{aligned}J_5 &= \sum_{j=0}^{k} C(j,k)\sum_{i=0}^{m+r-1}\frac{f^{(i)}(x)}{i!}\int_0^1 W_{d_jn}^{(r)}(t,x)(t-x)^i(g(t)-g(x))\,dt + o\left(n^{-(k+1)}\right) \\ &= J_7 + o\left(n^{-(k+1)}\right), \text{ say.}\end{aligned}$$

Since $g \in C_0^\infty$, therefore we can write

$$\begin{aligned} J_7 &= \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{m+r-1} \frac{f^{(i)}(x)}{i!} \sum_{p=1}^{m+r+1} \frac{g^{(p)}(x)}{p!} \int_0^1 W_{d_jn}^{(r)}(t,x)(t-x)^{i+p}\,dt \\ &+ \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{m+r-1} \frac{f^{(i)}(x)}{i!} \int_0^1 W_{d_jn}^{(r)}(t,x)\,\epsilon(t,x)(t-x)^{i+m+r+1}\,dt \\ &= J_8 + J_9, \text{ say,} \end{aligned}$$

where $\epsilon(t,x) \to 0$ as $t \to x$.
By Lemma 4 and Theorem 1, we obtain

$$\begin{aligned} J_8 &= \sum_{i=1}^{r} \frac{g^{(i)}(x) f^{(r-i)}(x)}{i!(r-i)!} r! + O\left(n^{-(k+1)}\right) \\ &= \sum_{i=1}^{r} \binom{r}{i} g^{(i)}(x) f^{(r-i)}(x) + O\left(n^{-(k+1)}\right), \end{aligned}$$

uniformly on $[z_2, w_2]$.
To estimate J_9, it is sufficient to treat it without linear combination. Let

$$J \equiv P_n^{(r)}(\epsilon(t,x)(t-x)^{i+m+r+1}; x).$$

By using Lemma 2 we have

$$\begin{aligned} |J| &\leq \sum_{\substack{2p+j\leq r \\ p,j\geq 0}} n^p \frac{|q_{p,j,r}(x)|}{x^r(1-x)^r} \sum_{\nu=1}^{n} |\nu - nx|^j \, p_{n,\nu}(x) \times \\ &\qquad \int_0^1 p_{n-1,\nu-1}(t) |\epsilon(t,x)| |t-x|^{i+m+r+1}\,dt \\ &+ \frac{(n+r-1)!}{(n-1)!}(1-x)^{-n-r} |\epsilon(0,x)| x^{i+m+r+1} \\ &= J_{10} + J_{11}, \text{ say.} \end{aligned}$$

Since $\epsilon(t,x) \to 0$ as $t \to x$, for a given $\epsilon' > 0$ we can find a $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon'$ whenever $0 < |t-x| < \delta$ and for $|t-x| \geq \delta$, $|\epsilon(t,x)| \leq K$ for some $K > 0$. Hence

$$\begin{aligned}
|J_{10}| &\leq \sum_{\substack{2p+j\leq r\\ p,j\geq 0}} n^p \frac{|q_{p,j,r}(x)|}{x^r(1-x)^r} \sum_{\nu=1}^{n} |k-nx|^j \, p_{n,\nu}(x) \times \\
& \quad \left[\epsilon \int\limits_{|t-x|<\delta} p_{n-1,\nu-1}(t)|t-x|^{i+m+r+1}\, dt + \frac{1}{\delta^s} \int\limits_{|t-x|\geq\delta} p_{n-1,\nu-1}(t) K|t-x|^s \, dt\right], \\
& \qquad \text{for any } s>0 \\
&= J_{12} + J_{13}, \text{ say.}
\end{aligned}$$

Let $C_1 = \sup_{\substack{2p+j\leq r\\ p,j\geq 0}} |q_{p,j,r}(x)| / x^r(1-x)^r$.
Applying Schwarz inequality for integration and then for summation and Lemma 4, 3 we have

$$\begin{aligned}
|J_{12}| &\leq C_1\epsilon' \sum_{\substack{2p+j\leq p\\ p,j\geq 0}} n^p \left(\sum_{\nu=1}^{n}(\nu-nx)^{2j}\, p_{n,\nu}(x)\right)^{1/2} \times \left(\int_0^1 p_{n-1,\nu-1}(t)dt\right)^{1/2} \\
& \quad \left(\sum_{\nu=1}^{n} p_{n,\nu}(x) \int\limits_{|t-x|<\delta} p_{n-1,\nu-1}(t)(t-x)^{2i+2m+2r+2}\, dt\right)^{1/2} \\
&\leq C_1\epsilon' \sum_{\substack{2p+j\leq r\\ p,j\geq 0}} n^p \; O(n^{j/2}) \; O(n^{-(i+m+r+1)/2}) \\
&= \epsilon' \; O(n^{-(i+m+1)/2}), \; i \in \mathbb{N}^0 \\
&= \epsilon' \; O(n^{-(m+1)/2}).
\end{aligned}$$

Next, again applying Schwarz inequality for integration and then for summation and Lemma 4, 3, on choosing s to be any positive integer $> m+r+1$, we have

$$\begin{aligned}
J_{13} &\leq C_1 \sum_{\substack{2p+j\leq r\\ p,j\geq 0}} n^{p+1} \sum_{\nu=1}^{n} p_{n,\nu}(x)|\nu-nx|^j \left(\int_0^1 p_{n-1,\nu-1}(t)(t-x)^{2s}dt\right)^{1/2} \\
&\leq C_1 \sum_{\substack{2p+j\leq r\\ p,j\geq 0}} n^{p} \left(\sum_{\nu=1}^{n} p_{n,\nu}(x)(\nu-nx)^{2j}\right)^{1/2} \times \\
&\quad \left(n\sum_{\nu=1}^{n} p_{n,\nu}(x)\int_0^1 p_{n-1,\nu-1}(t)(t-x)^{2s}dt\right)^{1/2} \\
&\leq C_1 \sum_{\substack{2p+j\leq r\\ p,j\geq 0}} n^{p}\, O(n^{j/2})\, O(n^{-s/2}) \\
&= O(n^{(r-s)/2}) \\
&= o(n^{-(m+1)/2}).
\end{aligned}$$

Combining the estimates of J_{12} and J_{13}, we get

$$J_{10} = \epsilon' O(n^{-(m+1)/2}) + o(n^{-(m+1)/2}, \text{ uniformly on } [z_2, w_2].$$

Clearly,

$$\begin{aligned}
J_{11} &= O(n^{-s}) \text{ (for any } s>0) \\
&= O(n^{-\tau/2}), \text{ uniformly on } [z_2, w_2].
\end{aligned}$$

Therefore,

$$J_9 = O(n^{-\tau/2}), \text{ uniformly on } [z_2, w_2].$$

Next, using the mean value theorem, Schwarz inequality for integration and then for summation and Lemma 4, 3 for any $\delta > 0$ we have

$$
\begin{aligned}
|J_6| &\leq \\
&\leq \sum_{j=0}^{k}|C(j,k)|\int_0^1\left|W_{d_jn}^{(r)}(t,x)\right| \\
&\qquad \left\{\frac{|f^{(m+r-1)}(\xi)-f^{(m+r-1)}(x)|}{(m+r-1)!}|t-x|^{m+r}|g'(\eta)|\chi_2(t)\right\}dt \\
&\leq M\|g'\|_{C[z_2,w_2]}\sum_{j=0}^{k}|C(j,k)|\int_0^1\left|W_{d_jn}^{(r)}(t,x)\right||t-x|^{1-\delta}||t-x|^{m+r}\chi_2(t)\,dt \\
&\leq M\|g'\|_{C[z_2,w_2]}\sum_{j=0}^{k}|C(j,k)|\left[d_j\sum_{\nu=1}^{n}|p_{d_jn}^{(r)}(x)|\int_0^1 p_{n-1,\nu-1}(t)|t-x|^{m+r+1-\delta}\chi_2(t)\,dt\right. \\
&+ \left.\frac{(d_jn+r-1)!}{(d_jn-1)!}(1-x)^{-d_jn-r}x^{m+r+1-\delta}\right] \\
. &= O(n^{(-(m+1-\delta)/2})+O(n^{-s}),\ \text{for any } s>0 \\
&= O(n^{-\tau/2}),\ \text{on choosing }\ 0<\delta\leq m+1-\tau(>0).
\end{aligned}
$$

Combining the above estimates, we get

$$\|M_n^{(r)}(fg,k,.)-(fg)^{(r)}\|_{C[z_2,w_2]}=O(n^{-\tau/2}).$$

Since supp$fg\subset(z_2,w_2)$ by Lemmas 5 and 7 it follows that
$(fg)^{(r)}\in \mathrm{Liz}(\beta,k+1;z_2,w_2)$. Since $g(x)=1$ on I_2, it follows that
$f^{(r)}\in \mathrm{Liz}(\beta,k+1;I_2)$.
This completes the proof of $(i)\to(ii)$.
The equivalence of (ii) and (iii) is well known [2].
The implication $(iii)\to(iv)$ follows from Theorem 2.
This completes the proof of the inverse theorem.

□

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Karunesh Singh
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667(Uttarakhand), India
email:*kksiitr.singh@gmail.com*

P.N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667(Uttarakhand), India
email: *pnappfma@gmail.com*