Acta Universitatis Apulensis No. 32/2012
ISSN: 1582-5329 pp. 285-291

# NEIGHBORHOOD PROPERTIES OF MULTIVALENT FUNCTIONS DEFINED USING AN INTEGRAL OPERATOR

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Abstract. In this paper, we introduce the generalized integral operator $J_p(\sigma,\lambda)$ and using this generalized integral operator, the new subclasses $\mathcal{H}^p_{n,m}(b,\sigma,\lambda)$, $\mathcal{L}^p_{n,m}(b,\sigma,\lambda;\mu)$, $\mathcal{H}^{p;\alpha}_{n,m}(b,\sigma,\lambda)$ and $\mathcal{L}^{p;\alpha}_{n,m}(b,\sigma,\lambda;\mu)$ of the class of multivalent functions denoted by $\mathcal{T}_p(n)$ are defined. Further for functions belonging to these classes, certain properties of neighborhoods of functions of complex order are studied.

2000 *Mathematics Subject Classification*: 30C45.

## 1.INTRODUCTION

Let $\mathcal{A}_p(n)$ be the class of normalized functions $f$ of the form

$$f(z)=z^p+\sum_{k=n+p}^{\infty}a_kz^k, \quad (n,\,p\in\mathbb{N}), \tag{1}$$

which are analytic and $p$ -valent in the open unit disc $\mathcal{U}=\{z\in\mathbb{C}\,:\,|z|<\,1\}$.
Let $\mathcal{T}_p(n)$ be the subclass of $\mathcal{A}_p(n)$ consisting functions $f$ of the form

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k, \quad (a_k\geq 0,\ n,\,p\in\mathbb{N}), \tag{2}$$

which are $p$ - valent in $\mathcal{U}$.

**Definition 1** *Let $\sigma,\ \lambda\in\mathbb{R},\ \sigma>0,\ \lambda>-p,\ p\in\mathbb{N}$ and $f\in\mathcal{A}_p(n)$, the integral operator $J_p(\sigma,\lambda)$ is defined as*

$$J_p(\sigma,\lambda)f(z)=\frac{(\lambda+p)^\sigma}{z^\lambda\Gamma(\sigma)}\int_0^z t^{\lambda-1}\left(\log\frac{z}{t}\right)^{\sigma-1}f(t)\,dt=z^p+\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^\sigma a_kz^k, \tag{3}$$

*where $\Gamma$ denotes the Gamma function.*

**Remark 1** *We observe that the operator* $J_1(\sigma,\lambda) \equiv P_\lambda^\sigma$ *introduced by Gao, Yuan and Srivastava [1],* $J_1(\sigma,1) \equiv I^\sigma$ *studied by Miller and Mocanu [4] and also* $J_1(\sigma,1) \equiv P^\sigma$ *introduced by Jung, Kim and Srivastava [3].*

For any function $f \in \mathcal{T}_{p(n)}$ and $\delta \geq 0$, the $(n,\delta)$ - neighborhood of $f$ is defined as,

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{T}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta \right\}. \quad (4)$$

For the function $h(z) = z^p$, $(p \in \mathbb{N})$ we have,

$$\mathcal{N}_{n,\delta}(h) = \left\{ g \in \mathcal{T}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k|b_k| \leq \delta \right\}. \quad (5)$$

The concept of neighborhoods was first introduced by Goodman [2] and then generalized by Ruscheweyh [8].

**Definition 2** *A function* $f \in \mathcal{T}_p(n)$ *is said to be in the class* $\mathcal{H}_{n,m}^p(b,\sigma,\lambda)$ *if*

$$\left| \frac{1}{b} \left( \frac{z\left(J_p(\sigma,\lambda)f(z)\right)^{(m+1)}}{\left(J_p(\sigma,\lambda)f(z)\right)^{(m)}} - (p-m) \right) \right| < 1, \quad (6)$$

*where,* $p \in \mathbb{N},\ m \in \mathbb{N}_0,\ \lambda > -p,\ \sigma > 0,\ p > m,\ b \in \mathbb{C} \setminus \{0\}$ *and* $z \in \mathcal{U}$.

**Definition 3** *A function* $f \in \mathcal{T}_p(n)$ *is said to be in the class* $\mathcal{L}_{n,m}^p(b,\sigma,\lambda;\mu)$ *if*

$$\left| \frac{1}{b} \left[ p(1-\mu) \left( \frac{J_p(\sigma,\lambda)f(z)}{z} \right)^{(m)} + \mu \left(J_p(\sigma,\lambda)f(z)\right)^{(m+1)} - (p-m) \right] \right| < p-m, \quad (7)$$

*where,* $p \in \mathbb{N},\ m \in \mathbb{N}_0,\ \lambda > -p,\ \sigma > 0,\ \mu \geq 0,\ p > m,\ b \in \mathbb{C} \setminus \{0\}$ *and* $z \in \mathcal{U}$.

## 2.Coefficient bounds

In this section, we obtain the coefficient inequalities for functions belonging to the classes $\mathcal{H}_{n,m}^p(b,\sigma,\lambda)$ and $\mathcal{L}_{n,m}^p(b,\sigma,\lambda;\mu)$.

**Theorem 2.1***Let* $f \in \mathcal{T}_p(n)$. *Then,* $f \in \mathcal{H}_{n,m}^p(b,\sigma,\lambda)$ *if and only if*

$$\sum_{k=n+p}^{\infty} \left( \frac{\lambda+p}{\lambda+k} \right)^{\sigma} \binom{k}{m} (k + |b| - p)\, a_k \leq |b| \binom{p}{m}. \quad (8)$$

*Proof.* Let $f \in \mathcal{H}^p_{n,m}(b,\sigma,\lambda)$. Then, by (6) and (7) we can write,

$$\Re\left\{\frac{\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^{\sigma}\binom{k}{m}(p-k)a_k z^{k-p}}{\binom{p}{m}-\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^{\sigma}\binom{k}{m}a_k z^{k-p}}\right\} > -|b|, \quad (z\in\mathcal{U}). \tag{9}$$

Taking $|z| = r,\ (0 \le r < 1)$ in ( 9), we see that the expression in the denominator on the Left Hand Side of ( 9), is positive for $r = 0$ and also for all $r,\ 0 \le r < 1$. Hence, by letting $r \mapsto 1^-$ through real values, expression ( 9) yields the desired condition ( 8). Conversely, by applying the hypothesis ( 8) and letting $|z| = 1$,we obtain,

$$\left|\frac{z\left(J_p(\sigma,\lambda)f(z)\right)^{(m+1)}}{\left(J_p(\sigma,\lambda)f(z)\right)^{(m)}}-(p-m)\right| = \left|\frac{\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^{\sigma}\binom{k}{m}(p-k)a_k z^{k-m}}{\binom{p}{m}z^{p-m}-\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^{\sigma}\binom{k}{m}a_k z^{k-m}}\right|$$

$$\le \frac{|b|\left[\binom{p}{m}-\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^{\sigma}\binom{k}{m}a_k\right]}{\binom{p}{m}-\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^{\sigma}\binom{k}{m}a_k} = |b|.$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{H}^p_{n,m}(b,\sigma,\lambda)$. Thus the proof is complete. On similar lines, we can prove the following Theorem.

**Theorem 2.2***A function $f \in \mathcal{L}^p_{n,m}(b,\sigma,\lambda;\mu)$ if and only if*

$$\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)^{\sigma}\binom{k-1}{m}\left[\mu(k-1)+1\right]a_k \le (p-m)\left[\frac{|b|-1}{m!}+\binom{p}{m}\right]. \tag{10}$$

## 3.Inclusion relationships involving $(n,\delta)$ - neighborhoods

In this section, we study inclusion relationship for the functions belonging to the classes $\mathcal{H}^p_{n,m}(b,\sigma,\lambda)$ and $\mathcal{L}^p_{n,m}(b,\sigma,\lambda;\mu)$.

**Theorem 3.1** *If*

$$\delta = \frac{(n+p)|b|\binom{p}{m}}{(n+|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}}, \quad (p>|b|), \tag{11}$$

*then* $\mathcal{H}^{p}_{n,m}(b,\sigma,\lambda) \subset \mathcal{N}_{n,\delta}(h)$.

*Proof.* Let $f \in \mathcal{H}^{p}_{n,m}(b,\sigma,\lambda)$. By Theorem 2.1 we have,

$$(n+|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k \leq |b|\binom{p}{m}$$

which implies,

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|\binom{p}{m}}{(n+|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}}. \tag{12}$$

Using ( 8) and( 12), we have,

$$\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}\sum_{k=n+p}^{\infty} ka_k \leq |b|\binom{p}{m} + (p-|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k$$

$$\leq |b|\binom{p}{m} + (p-|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}\frac{|b|\binom{p}{m}}{(n+|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}}$$

$$= |b|\binom{p}{m}\frac{n+p}{n+|b|}.$$

That is,

$$\sum_{k=n+p}^{\infty} ka_k \leq \frac{|b|(n+p)\binom{p}{m}}{(n+|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}} = \delta, \quad (p>|b|).$$

Thus, by the definition given by ( 7), $f \in \mathcal{N}_{n,\delta}(h)$. This completes the proof. Similarly, we prove the following Theorem.

**Theorem 3.2** *If*

$$\delta = \frac{(p-m)(n+p)\left[\frac{|b|-1}{m!} + \binom{p}{m}\right]}{[\mu(n+p-1)+1]\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}}, \quad (\mu>1) \tag{13}$$

*then* $\mathcal{L}^{p}_{n,m}(b,\sigma,\lambda;\mu) \subset \mathcal{N}_{n,\delta}(h)$.

## 4.Further Neighborhood Properties

Now, we determine the neighborhood properties for functions belonging to the classes $\mathcal{H}^{p,\alpha}_{n,m}(b,\sigma,\lambda)$ and $\mathcal{L}^{p,\alpha}_{n,m}(b,\sigma,\lambda;\mu)$.
For $0 \leq \alpha < p$ and $z \in \mathcal{U}$, a function $f$ is said to be in the class $\mathcal{H}^{p,\alpha}_{n,m}(b,\sigma,\lambda)$ if there exists a function $g \in \mathcal{H}^{p}_{n,m}(b,\sigma,\lambda)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha. \tag{14}$$

For $0 \leq \alpha < p$ and $z \in \mathcal{U}$, a function $f$ is said to be in the class $\mathcal{L}^{p,\alpha}_{n,m}(b,\sigma,\lambda;\mu)$ if there exists a function $g \in \mathcal{L}^{p}_{n,m}(b,\sigma,\lambda;\mu)$ such that the inequality ( 14) holds true.

**Theorem 4.1**$\mathcal{N}_{n,\delta}(g) \subset \mathcal{H}_{n,m}$. If $g \in \mathcal{H}^{p}_{n,m}(b,\sigma,\lambda)$ and

$$\alpha = p - \frac{\delta(n+|b|)\left(\dfrac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}}{(n+p)\left[(n+|b|)\left(\dfrac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m} - |b|\binom{p}{m}\right]}, \tag{15}$$

$\alpha(b,\sigma,\lambda)$.

*Proof. Let* $f \in \mathcal{N}_{n,\delta}(g)$. *Then,*

$$\sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta, \tag{16}$$

*which yields the coefficient inequality,*

$$\sum_{k=n+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+p}, \quad (n \in \mathbb{N}). \tag{17}$$

*Since* $g \in \mathcal{H}^{p}_{n,m}(b,\sigma,\lambda)$ *by ( 12), we have,*

$$\sum_{k=n+p}^{\infty} b_k \leq \frac{|b|\binom{p}{m}}{(n+|b|)\left(\dfrac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}} \tag{18}$$

*so that,*

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum\limits_{k=n+p}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=n+p}^{\infty}b_k}$$

$$\leq \frac{\delta}{n+p}\,\frac{(n+|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}}{\left[(n+|b|)\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p}{m}-|b|\binom{p}{m}\right]}$$

$$= p-\alpha.$$

*Thus, by definition,* $f \in \mathcal{H}^{p;\alpha}_{n,m}(b,\sigma,\lambda)$ *for* $\alpha$ *given by* ( 15). *Thus the proof is complete.*

On similar lines, we prove the following theorem.

**Theorem 4.2***If* $g \in \mathcal{L}^{p}_{n,m}(b,\sigma,\lambda;\mu)$ *and*

$$\alpha = p - \frac{\delta[\mu(n+p-1)+1]\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p-1}{m}}{(n+p)\left[\{\mu(n+p-1)+1\}\left(\frac{\lambda+p}{\lambda+n+p}\right)^{\sigma}\binom{n+p-1}{m}-(p-m)\left(\frac{|b|-1}{m!}+\binom{p}{m}\right)\right]}, \tag{19}$$

*then* $\mathcal{N}_{n,\delta}(g) \subset \mathcal{L}^{p,\alpha}_{n,m}(b,\sigma,\lambda;\mu)$.

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