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SOME RESULTS RELATED TO TOPOLOGICAL GROUPS VIA IDEAL TOPOLOGICAL SPACES

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Abstract. An ideal on a set X is a nonempty collection of subsets of X with heredity property which is also closed finite unions. The concept of $\Gamma_\delta : \mathcal{P}(X) \to \tau$ defined as follows for every $A \in X$, $\Gamma_\delta(A) = \{x \in X$: there exists a $U \in \tau^\delta(x)$ such that $U - A \in \mathcal{I}\}$, was introduced by Al-Omari and Hatir [1]. In this paper, we introduce and study $\delta*$-homeomorphism and Γ_δ-homeomorphism. Also we give some application to topological groups using δ-open function and Γ_δ-operator.

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1. Introduction and preliminaries

Ideals in topological spaces have been considered since 1930. This topic has won its importance by Vaidyanathaswamy [13]. In [2] Janković and Hamlett investigated further properties of ideal topological space. In this paper, we investigated δ-local function and its properties in ideal topological space. Moreover, the relationships other local functions [2, 7, 8] are investigated.

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y), always mean topological spaces on which no separation axiom is assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively. A subset A of a space (X, τ) is said to be regular open (resp. regular closed) [12] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is called δ-open [12] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of a δ-open set is called δ-closed. A point $x \in X$ is called a δ-cluster point of A if $Int(Cl(U)) \cap A \neq \phi$ for each open set V containing x.

The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta Cl(A)$. The δ-interior of A is the union of all regular open sets of X contained in A and it is denoted by $\delta Int(A)$. A is δ-open if $\delta Int(A) = A$. δ-open sets forms a topology τ^{δ}. Actually τ^{δ} is the same as the collection of all δ-open sets of (X, τ) and is denoted by $\delta O(X)$. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies $i)$ $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$, $ii)$ $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$ called a local function [2, 7] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. We simply write A^* instead of $A^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X. The hypothesis $X = X^*$ [6] is equivalent to the hypothesis $\tau \cap \mathcal{I} = \phi$. For every ideal topological space, there exists a topology $\tau^*(\mathcal{I})$ or briefly τ^*, finer than τ, generated by $\beta(\mathcal{I}, \tau) = \{U - I : U \in \tau$ and $I \in \mathcal{I}\}$, but in general $\beta(\mathcal{I}, \tau)$ is not always a topology [2]. Additionally, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(\mathcal{I})$. If $\mathcal{I}$ is an ideal on X then $(X, \tau, \mathcal{I})$ is called an ideal topological space. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say that the topology τ is *compatible* with the ideal $\mathcal{I}$, denoted $\tau \sim \mathcal{I}$, if the following hold for every $A \subset X$, if for every $x \in A$ there exists a $U \in \tau$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$ [2]. Quite recently, Al-Omari and Hatir, [1] defined the $\Gamma_\delta : \mathcal{P}(X) \to \tau$ as follows for every $A \in X$, $\Gamma_\delta(A) = \{x \in X :$ there exists a $U \in \tau^{\delta}(x)$ such that $U - A \in \mathcal{I}\}$. In this paper, we introduce and study δ^*-homeomorphism and Γ_δ-homeomorphism. Also we give some application to topological groups using δ-open function and Γ_δ-operator. In [10], Newcomb defines $A = B$ [mod $\mathcal{I}$] if $(A - B) \cup (B - A) \in \mathcal{I}$ and observes that $=$ [mod $\mathcal{I}$] is an equivalence relation.

Definition 1. *[1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a Baire set with respect to τ^{δ} and $\mathcal{I}$, denoted $A \in \mathcal{B}_r(X, \tau, \mathcal{I})$, if there exists a δ-open set U such that $A = U$ [mod $\mathcal{I}$]. Let $\mathcal{U}(X, \tau, \mathcal{I})$ be denoted $\{A \subseteq X :$ there exists $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.*

2. δ-local functions and Γ_δ-operator

Let $(X, \tau, \mathcal{I})$ an ideal topological space and A a subset of X. Then $A^{\delta *}(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \delta O(X, x)\}$ is called δ-local function [5] of A with respect to $\mathcal{I}$ and τ, where $\delta O(X, x) = \{U \in \delta O(X) : x \in U\}$. We denote simply $A^{\delta *}$ for $A^{\delta *}(\mathcal{I}, \tau)$.

Remark 1. *[5]*

1. *The simplest ideals are* $\{\phi\}$ *and* $\mathcal{P}(X) = \{A : A \subset X\}$. *It can be deduce that* $A^{\delta*}(\{\phi\}) = \delta Cl(A) \neq Cl(A)$ *and* $A^{\delta*}(\mathcal{P}(X)) = \phi$ *for every* $A \subset X$.

2. *If* $A \in \mathcal{I}$, *then* $A^{\delta*} = \phi$.

3. *Neither* $A \subset A^{\delta*}$ *nor* $A^{\delta*} \subset A$ *in general.*

Theorem 1. *[5] Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space, then the following properties are equivalent:*

1. $\tau^{\delta} \cap \mathcal{I} = \phi$;

2. *If* $I \in \mathcal{I}$, *then* $\delta Int(I) = \phi$;

3. *For every* $G \in \tau^{\delta}$, $G \subseteq G^{\delta*}$;

4. $X = X^{\delta*}$.

Theorem 2. *[5] Let* $(X, \tau, \mathcal{I})$ *an ideal topological space and* A, B *subsets of* X. *Then for* δ*-local functions the following properties hold:*

1. *If* $A \subset B$, *then* $A^{\delta*} \subset B^{\delta*}$,

2. $A^{\delta*} = \delta Cl(A^{\delta*}) \subset \delta Cl(A)$ *and* $A^{\delta*}$ *is* δ*-closed,*

3. $(A^{\delta*})^{\delta*} \subset A^{\delta*}$,

4. $(A \cup B)^{\delta*} = A^{\delta*} \cup B^{\delta*}$,

5. $A^{\delta*} - B^{\delta*} = (A - B)^{\delta*} - B^{\delta*} \subset (A - B)^{\delta*}$,

6. *If* $U \in \tau^{\delta}$, *then* $U \cap A^{\delta*} = U \cap (U \cap A)^{\delta*} \subset (U \cap A)^{\delta*}$,

7. *If* $U \in \mathcal{I}$, *then* $(A - U)^{\delta*} = A^{\delta*} = (A \cup U)^{\delta*}$,

8. *If* $A \subseteq A^{\delta*}$, *then* $A^{\delta*} = \delta Cl(A^{\delta*}) = \delta Cl(A)$.

Theorem 3. *[5] Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space, then the following are equivalent:*

1. $\tau \sim^{\delta} \mathcal{I}$,
2. If a subset A of X has a cover of δ-open sets each of whose intersection with A is in $\mathcal{I}$, then A is in $\mathcal{I}$,
3. For every $A \subset X$, if $A \cap A^{\delta *} = \phi$, $A \in \mathcal{I}$,
4. For every $A \subset X$, if $A - A^{\delta *} \in \mathcal{I}$,
5. For every $A \subset X$, if A contains no nonempty subset B with $B \subset B^{\delta *}$, then $A \in \mathcal{I}$.

Let us denote $\beta(\mathcal{I}, \tau) = \{V - I_o \ : \ V \in \delta O(X),\ I_o \in \mathcal{I}\}$, simplicity $\beta(\mathcal{I}, \tau)$ for β.

Theorem 4. *[5] Let (X, τ) be a space, $\mathcal{I}$ an ideal on X . Then β is a basis for $\tau^{\delta *}$.*

Theorem 5. *[5] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If τ is δ-compatible with $\mathcal{I}$, then the following equivalent properties hold:*

1. *For every $A \subseteq X$, $A \cap A^{\delta *} = \phi$ implies that $A^{\delta *} = \phi$.*
2. *For every $A \subseteq X$, $(A - A^{\delta *})^{\delta *} = \phi$.*
3. *For every $A \subseteq X$, $(A \cap A^{\delta *})^{\delta *} = A^{\delta *}$.*

Theorem 6. *[1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties hold:*

1. *If $A \subseteq X$, then $\Gamma_\delta(A)$ is δ-open.*
2. *If $A \subseteq B$, then $\Gamma_\delta(A) \subseteq \Gamma_\delta(B)$.*
3. *If $A, B \in X$, then $\Gamma_\delta(A \cap B) = \Gamma_\delta(A) \cap \Gamma_\delta(B)$.*
4. *If $U \in \tau^{\delta *}$, then $U \subseteq \Gamma_\delta(U)$.*
5. *If $A \subseteq X$, then $\Gamma_\delta(A) \subseteq \Gamma_\delta(\Gamma_\delta(A))$.*
6. *If $A \subseteq X$, then $\Gamma_\delta(A) = \Gamma_\delta(\Gamma_\delta(A))$ if and only if $(X - A)^{\delta *} = ((X - A)^{\delta *})^{\delta *}$.*

7. *If* $A \in \mathcal{I}$, *then* $\Gamma_\delta(A) = X - X^{\delta*}$.

8. *If* $A \subseteq X$, *then* $A \cap \Gamma_\delta(A) = Int^{\delta*}(A)$.

9. *If* $A \subseteq X$, $I \in \mathcal{I}$, *then* $\Gamma_\delta(A - I) = \Gamma_\delta(A)$.

10. *If* $A \subseteq X$, $I \in \mathcal{I}$, *then* $\Gamma_\delta(A \cup I) = \Gamma_\delta(A)$.

11. *If* $(A - B) \cup (B - A) \in \mathcal{I}$, *then* $\Gamma_\delta(A) = \Gamma_\delta(B)$.

Theorem 7. *[1] Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space with* $\tau \sim^\delta \mathcal{I}$. *Then* $\Gamma_\delta(A) = \cup\{\Gamma_\delta(U) : U \in \tau^\delta, \Gamma_\delta(U) - A \in \mathcal{I}\}$.

Proposition 1. *[1] Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space with* $\tau \sim^\delta \mathcal{I}$, $A \subseteq X$. *If* N *is a nonempty* δ*-open subset of* $A^{\delta*} \cap \Gamma_\delta(A)$, *then* $N - A \in \mathcal{I}$ *and* $N \cap A \notin \mathcal{I}$.

Theorem 8. *[1] Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. Then* $\tau \sim^\delta \mathcal{I}$ *if and only if* $\Gamma_\delta(A) - A \in \mathcal{I}$ *for every* $A \subseteq X$.

Proposition 2. *[1] Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space with* $\tau^\delta \cap \mathcal{I} = \phi$. *The following properties are equivalent:*

1. $A \in \mathcal{U}(X, \tau, \mathcal{I})$;

2. $\Gamma_\delta(A) \cap \delta Int(A^{\delta*}) \neq \phi$;

3. $\Gamma_\delta(A) \cap A^{\delta*} \neq \phi$;

4. $\Gamma_\delta(A) \neq \phi$;

5. $Int^{\delta*}(A) \neq \phi$;

6. *There exists* $N \in \tau^\delta - \{\phi\}$ *such that* $N - A \in \mathcal{I}$ *and* $N \cap A \notin \mathcal{I}$.

3. $\delta*$-HOMEOMORPHISMS

Given an ideal topological space $(X, \tau, \mathcal{I})$ a topology denoted by $\langle \Gamma_\delta(\tau) \rangle$, coarser than τ^δ is generated by the basis $\Gamma_\delta(\tau) = \{\Gamma_\delta(U) : U \in \tau^\delta\}$

Definition 2. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be an ideal topological spaces. A bijection $f:(X,\tau,\mathcal{I})\rightarrow(Y,\sigma,\mathcal{J})$ is called*

1. *$\delta*$-homeomorphism if $f:(X,\tau^{\delta*})\rightarrow(Y,\sigma^{\delta*})$ is a homeomorphism.*

2. *Γ_δ-homeomorphism if $f:(X,\Gamma_\delta(\tau))\rightarrow(Y,\Gamma_\delta(\sigma))$ is a homeomorphism.*

Definition 3. *A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is called*

1. *δ-continuous [11] if the inverse image of δ-open set is δ-open.*

2. *δ-open if the image of δ-open set is δ-open.*

Theorem 9. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be an ideal topological spaces with $f:(X,\tau)\rightarrow(Y,\Gamma_\delta(\sigma))$ is a δ-continuous injection, $\sigma\sim^{\delta}\mathcal{J}$ and $f^{-1}(\mathcal{J})\subseteq\mathcal{I}$. Then $\Gamma_\delta(f(A))\subseteq f(\Gamma_\delta(A))$ for every $A\subseteq X$.*

Proof. Let $y\in\Gamma_\delta(f(A))$ where $A\subseteq X$. Then by Theorem 7, there exists $V\in\sigma^\delta$ such that $y\in\Gamma_\delta(V)$ and $\Gamma_\delta(V)-f(A)\in\mathcal{J}$. Now we have $f^{-1}(\Gamma_\delta(V))\in\tau^\delta(f^{-1}(y))$ with $f^{-1}\left[\Gamma_\delta(V)-f(A)\right]\in\mathcal{I}$, then $f^{-1}\left[\Gamma_\delta(V)\right]-A\in\mathcal{I}$ and $f^{-1}(y)\in\Gamma_\delta(A)$ and hence $y\in f(\Gamma_\delta(A))$, and the proof is complete. □

Theorem 10. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be an ideal topological spaces with $f:(X,\Gamma_\delta(\tau))\rightarrow(Y,\sigma,\mathcal{J})$ an δ-open bijective, $\tau\sim^{\delta}\mathcal{I}$ and $f(\mathcal{I})\subseteq\mathcal{J}$. Then $f(\Gamma_\delta(A))\subseteq\Gamma_\delta(f(A))$ for every $A\subseteq X$.*

Proof. Let $A\subseteq X$ and let $y\in f(\Gamma_\delta(A))$. Then $f^{-1}(y)\in\Gamma_\delta(A)$ and there exists $V\in\tau^\delta$ such that $f^{-1}(y)\in\Gamma_\delta(V)$ and $\Gamma_\delta(V)-A\in\mathcal{I}$ by Theorem 7. Now $f(\Gamma_\delta(V))\in\sigma^\delta(y)$ and $f(\Gamma_\delta(V))-f(A)=f\left[\Gamma_\delta(V)-A\right]\in f(\mathcal{I})\subseteq\mathcal{J}$. Thus $y\in\Gamma_\delta(f(A))$, and the proof is complete. □

Theorem 11. *Let $f:(X,\tau,\mathcal{I})\rightarrow(Y,\sigma,\mathcal{J})$ be a bijection with $f(\mathcal{I})=\mathcal{J}$. Then the following properties are equivalent:*

1. *f is $\delta*$-homeomorphism;*

2. *$f(A^{\delta*})=[f(A)]^{\delta*}$ for every $A\subseteq X$;*

3. $f(\Gamma_\delta(A)) = \Gamma_\delta(f(A))$ *for every* $A \subseteq X$*.*

Proof. (1) $\Rightarrow$ (2) Let $A \subseteq X$. Assume $y \notin f(A^{\delta *})$. This implies that $f^{-1}(y) \notin A^{\delta *}$, and hence there exists $U \in \tau^{\delta}(f^{-1}(y))$ such that $U \cap A \in \mathcal{I}$. Consequently $f(U) \in \sigma^{\delta *}(y)$ and $f(U) \cap f(A) \in \mathcal{J}$, then $y \notin [f(A)]^{\delta *}(\mathcal{J}, \sigma^{\delta *}) = [f(A)]^{\delta *}(\mathcal{J}, \sigma)$. Thus $[f(A)]^{\delta *} \subseteq f(A^{\delta *})$. Now assume $y \notin [f(A)]^{\delta *}$. This implies there exists a $V \in \sigma^{\delta *}(y)$ such that $V \cap f(A) \in \mathcal{J}$, then $f^{-1}(V) \in \tau^{\delta *}(f^{-1}(y))$ and $f^{-1}(V) \cap A \in \mathcal{I}$. Thus $f^{-1}(y) \notin A^{\delta *}(\mathcal{I}, \tau^{\delta *}) = A^{\delta *}(\mathcal{I}, \tau^{\delta})$ and $y \notin f(A^{\delta *})$. Hence $f(A^{\delta *}) \subseteq [f(A)]^{\delta *}$ and $f(A^{\delta *}) = [f(A)]^{\delta *}$.
(2) $\Rightarrow$ (3) Let $A \subseteq X$. Then $f(\Gamma_\delta(A)) = f[X - (X - A)^{\delta *}] = Y - f(X - A)^{\delta *} = Y - [Y - f(A)]^{\delta *} = \Gamma_\delta(f(A))$.
(3) $\Rightarrow$ (1) Let $U \in \tau^{\delta *}$. Then $U \subseteq \Gamma_\delta(U)$ by Theorem 6 and $f(U) \subseteq f(\Gamma_\delta(U)) = \Gamma_\delta(f(U))$. This shows that $f(U) \in \sigma^{\delta *}$ and hence $f : (X, \tau^{\delta *}) \to (Y, \sigma^{\delta *})$ is $\tau^{\delta *}$-open. Similarly, $f^{-1} : (Y, \sigma^{\delta *}) \to (X, \tau^{\delta *})$ is $\sigma^{\delta *}$-open and, f is $\delta *$-homeomorphism. □

Theorem 12. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space, then* $\langle \Gamma_\delta(\tau^{\delta *}) \rangle = \langle \Gamma_\delta(\tau^{\delta}) \rangle$*.*

Proof. Note that for every $U \in \tau^{\delta}$ and for every $I \in \mathcal{I}$, we have $\Gamma_\delta(U - I) = \Gamma_\delta(U)$. Consequently, $\Gamma_\delta(\beta) = \Gamma_\delta(\tau^{\delta})$ and $\langle \Gamma_\delta(\beta) \rangle = \langle \Gamma_\delta(\tau^{\delta}) \rangle$. It follows directly from Theorem 11 of [4] that $\langle \Gamma_\delta(\beta) \rangle = \langle \Gamma_\delta(\tau^{\delta *}) \rangle$, hence the theorem is proved. □

Theorem 13. *Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *be a bijection with* $f(\mathcal{I}) = \mathcal{J}$*. Then the following are hold:*

1. *If* f *is a* $\delta *$*-homeomorphism, then* f *is a* Γ_δ*-homeomorphism.*
2. *If* $\tau \sim^{\delta} \mathcal{I}$ *and* $\sigma \sim^{\delta} \mathcal{J}$ *and* f *is a* Γ_δ*-homeomorphism, then* f *is a* $\delta *$*-homeomorphism.*

Proof. (1) Assume $f : (X, \tau^{\delta *}) \to (Y, \sigma^{\delta *})$ is a $\delta *$-homeomorphism, and let $\Gamma_\delta(U)$ be a basic open set in $\langle \Gamma_\delta(\tau^{\delta}) \rangle$ with $U \in \tau^{\delta}$. Then $f(\Gamma_\delta(U)) = \Gamma_\delta(f(U))$ by Theorem 11. Then $f(\Gamma_\delta(U)) \in \Gamma_\delta(\sigma^{\delta *})$, but $\langle \Gamma_\delta(\tau^{\delta *}) \rangle = \langle \Gamma_\delta(\tau^{\delta}) \rangle$ by Theorem 12. Thus $f : (X, \Gamma_\delta(\tau)) \to (Y, \Gamma_\delta(\sigma))$ is δ-open. Similarly, $f^{-1} : (Y, \Gamma_\delta(\sigma)) \to (X, \Gamma_\delta(\tau))$ is δ-open and f is Γ_δ-homeomorphism.
(2) Assume f is a Γ_δ-homeomorphism, then $f(\Gamma_\delta(A)) = \Gamma_\delta(f(A))$ for every $A \subseteq X$ by Theorems 9 and 10. Thus f is a $\delta *$-homeomorphism by Theorem 11. □

4. Some results related to topological groups

Given a topological group $(X, \tau, .)$ and an ideal $\mathcal{I}$ on X, denoted $(X, \tau, \mathcal{I}, .)$ and $x \in X$, we denote by $x\mathcal{I} = \{xI : I \in \mathcal{I}\}$. We will say $\mathcal{I}$ is left translation invariant if for every $\in X$ we have $x\mathcal{I} \subseteq \mathcal{I}$. Observe that if $\mathcal{I}$ is left translation invariant then $x\mathcal{I} = \mathcal{I}$ for every $x \in X$. We defined $\mathcal{I}$ to be right translation invariant if and only if $\mathcal{I}x = \mathcal{I}$ for every $x \in X$ [3].

Lemma 1. *Let (X, τ) and (X, σ) be two topological spaces and $\mathscr{F}$ be a collection of δ-open mappings from X to Y. Let $U \in \tau^\delta - \phi$ and $\phi \neq A \subseteq U$. If $f(U) \in \mathscr{F}(A) = \{f(A) : f \in \mathscr{F}\}$ for every $f \in \mathscr{F}$, Then $\mathscr{F}(A) \in \sigma^\delta - \phi$.*

Proof. Let $y \in \mathscr{F}(A)$, then there exist $f \in \mathscr{F}$ such that $y \in f(A)$. Now, $A \subseteq U$, then $f(A) \subseteq f(U)$ and $y \in f(U)$. Then $f(U)$ is δ-open in (Y, σ) (as f is δ-open map). So there exists $V \in \sigma^\delta(y)$ such that $y \in V \subseteq f(U) \subseteq \mathscr{F}(A)$. So $\mathscr{F}(A) \in \sigma^\delta - \phi$. □

Theorem 14. *Let (X, τ) and (X, σ) be two topological spaces and $\mathcal{I}$ be an ideal (X, τ) with $\tau \sim^\delta \mathcal{I}$ and $\tau^\delta \cap \mathcal{I} = \phi$. Moreover, let $U \in \tau^\delta - \phi$, $A \subseteq X$, $U \subseteq A^{\delta*} \cap \Gamma_\delta(A)$ and $\mathscr{F}$ be a non-empty collection of δ-open mappings from X to Y. Suppose $y \in \mathscr{F}(U) \Rightarrow U \cap \mathscr{F}^{-1}(y) \notin \mathcal{I}$, where $\mathscr{F}^{-1}(y) = \cup\{f^{-1}(y) : f \in \mathscr{F}\}$. Then $\mathscr{F}(U \cap A) \in \sigma^\delta - \phi$.*

Proof. Since U is a non-empty δ-open set contained in $A^{\delta*} \cap \Gamma_\delta(A)$ and $\tau \sim^\delta \mathcal{I}$, by Proposition 1 it follows that $U - A \in \mathcal{I}$ and $U \cap A \notin \mathcal{I}$. For any $y \in \mathscr{F}(U)$, $U \cap \mathscr{F}^{-1}(y) \notin \mathcal{I}$ (by hypothesis) and we have $U \cap \mathscr{F}^{-1}(y) = U \cap \mathscr{F}^{-1}(y) \cap (A \cup A^c) = [U \cap \mathscr{F}^{-1}(y) \cap A] \cup [U \cap \mathscr{F}^{-1}(y) \cap A^c] \subseteq [U \cup \mathscr{F}^{-1}(y) \cap A] \cup (U - A)$ (where A^c = complement of A). Since $U \cap \mathscr{F}^{-1}(y) \notin \mathcal{I}$ and $U - A \in \mathcal{I}$, we have $U \cap \mathscr{F}^{-1}(y) \cap A \notin \mathcal{I}$. Then for any $y \in \mathscr{F}(U)$, $U \cap \mathscr{F}^{-1}(y) \cap A \neq \phi$. Now for a given $f \in \mathscr{F}$, $z \in f(U) \Rightarrow z \in \mathscr{F}(U)$, then there exist $x \in U \cap A$ and $x \in g^{-1}(z)$ for some $g \in \mathscr{F}$, where $z = g(x) \Rightarrow z \in g(U \cap A$, and $z \in \mathscr{F}(U \cap A)$. Hence $f(U) \subseteq \mathscr{F}(U \cap A)$, for all $f \in \mathscr{F}$. Then $\mathscr{F}(U \cap A) \in \sigma^\delta - \phi$ by Lemma 1. □

Lemma 2. *Let $\mathcal{I}$ be an ideal space on a topological group $(X, \tau, .)$ such that $\mathcal{I}$ is left or right translation invariant and $\tau \sim^\delta \mathcal{I}$. Then $\mathcal{I} \cap \tau^\delta = \phi$.*

Proof. Since $X \notin \mathcal{I}$ and $\tau \sim^\delta \mathcal{I}$, by Theorem 3 there exist $x \in X$ such that for all $U \in \tau^\delta(x)$, $U = U \cap X \notin \mathcal{I}$ .....(1)
Let $V \in \mathcal{I} \cap \tau^\delta$. If $V = \phi$ we have nothing to show. Suppose $V \neq \phi$. Without loss of

generality we may assume that $e \in V$ (e denoted the identity of X). For $y \in V$ then $y^{-1}V \in \tau^{\delta}$ and $y^{-1}V \in y^{-1}\mathcal{I}$ so that $y^{-1}V \in \mathcal{I}$ where $e \in y^{-1}V$. Thus $xV \in \tau^{\delta}$ and $xV \in x\mathcal{I}$ and hence $xV \in \mathcal{I}$. Thus $xV \in \tau^{\delta} \cap \mathcal{I}$, where xV is a neighbourhood of x, which is contradicting (1) and hence $\mathcal{I} \cap \tau^{\delta} = \phi$. $\square$

Lemma 3. *Let $\mathcal{I}$ be a left (right) translation invariant ideal on a topological group $(X, \tau, .)$ and $x \in X$. Then for any $A \subseteq X$ the following hold:*

1. $x\Gamma_{\delta}(A) = \Gamma_{\delta}(xA)$. *(resp.* $\Gamma_{\delta}(A)x = \Gamma_{\delta}(Ax)$*),*
2. $xA^{\delta *} = (xA)^{\delta *}$ *(resp.* $A^{\delta *}x = (Ax)^{\delta *}$*).*

Proof. We assume that $\mathcal{I}$ is left translation invariant, the proof for the case when $\mathcal{I}$ is right translation invariant would be similar.
(1) We first note that for any two subsets A and B of X, $x(A - B) = xA - xB$. In fact, $y \in x(A - B)$, then $y = xt$, for some $t \in A - B$. Now $t \in A$ then $xt \in xA$. But $xt \in xB \Rightarrow xt = xb$ for some $b \in B \Rightarrow t = b \in B$ a contradiction. So $y = xt \in xA - xB$. Again, $y \in xA - xB \Rightarrow y \in xA$ and $y \notin xB \Rightarrow y = xa$ for some $a \in A$ and $xa \notin xB \Rightarrow a \notin B \Rightarrow y = xa$, where $a \in A - B \Rightarrow y \in x(A - B)$.
Now, $y \in \Gamma_{\delta}(A) \Rightarrow y \in xU$ for some $U \in \tau^{\delta}$ with $U - A \in \mathcal{I}$. Then $xU = V \in \tau^{\delta}$ and $x(U - A) = xU - xA \in \mathcal{I}$ where $xU \in \tau^{\delta}$. Then $y \in V$, where $V \in \tau^{\delta}$ and $V - xA \in \mathcal{I} \Rightarrow y \in \cup\cup \{V \in \tau^{\delta} : V - xA \in \mathcal{I}\} = \Gamma_{\delta}(xA)$. Thus $x\Gamma_{\delta}(A) \subseteq \Gamma_{\delta}(xA)$.
Conversely, let $y \in \Gamma_{\delta}(xA) = \cup\{U \in \tau^{\delta} : U - xA \in \mathcal{I}\} \Rightarrow y \in U \in \tau^{\delta}$, where $U - xA \in \mathcal{I}$. Put $V = x^{-1}U$. Then $V \in \tau^{\delta}$. Now $x^{-1}y \in V$ and $V - A = x^{-1}U - A = x^{-1}(U - xA) \in \mathcal{I} \Rightarrow x^{-1}y \in \Gamma_{\delta}(A) \Rightarrow y \in x\Gamma_{\delta}(A)$. Thus $\Gamma_{\delta}(xA) \subseteq x\Gamma_{\delta}(A)$ and hence $x\Gamma_{\delta}(A) = \Gamma_{\delta}(xA)$
(2) In view of (1) $x\Gamma_{\delta}(X - A) = \Gamma_{\delta}(x(X - A))$, then $x[X - A^{\delta *}] = X - (xA)^{\delta *}$ and $X - xA^{\delta *} = X - (xA)^{\delta *}$ thus $xA^{\delta *} = (xA)^{\delta *}$. $\square$

Theorem 15. *Let $(X, \tau, .)$ be a topological group and $\mathcal{I}$ be an ideal on X such that $\tau \sim^{\delta} \mathcal{I}$. Let $P \in \mathcal{U}(X, \tau, \mathcal{I})$ and $Q \in \mathcal{P}(X) - \mathcal{I}$. Let $U, V \in \tau^{\delta}$ such that $U \cap Q^{\delta *} \neq \phi$, $V \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P) \neq \phi$. If $A = U \cap Q \cap Q^{\delta *}$ and $B = V \cap \delta Int(P^{\delta *}) \cap P \cap \Gamma_{\delta}(P)$ then the following hold:*

1. *If $\mathcal{I}$ is right translation invariant, then $A^{-1}B$ is a non-empty δ-open set contained in $Q^{-1}P$.*

2. *If $\mathcal{I}$ is left translation invariant, then BA^{-1} is a non-empty δ-open set contained in PQ^{-1}.*

Proof. (1) Since X is a topological group, $\tau \sim^{\delta} \mathcal{I}$ and $\mathcal{I}$ is right translation invariant, we have by Lemma 2, $\mathcal{I} \cap \tau^{\delta} = \phi$. Now by Theorem 2 $(U \cap Q \cap Q^{\delta *})^{\delta *} \subseteq (U \cap Q)^{\delta *}$ and by Theorem 5 we get $(U \cap Q \cap (U \cap Q)^{\delta *})^{\delta *} = (U \cap Q)^{\delta *}$. Hence $(U \cap Q \cap Q^{\delta *})^{\delta *} = (U \cap Q)^{\delta *}$ ....(1)
Thus by Theorem 2 we have $U \cap Q^{\delta *} = U \cap (U \cap Q)^{\delta *} \subseteq (U \cap Q)^{\delta *} = (U \cap Q \cap Q^{\delta *})^{\delta *}$ by (1). Since $U \cap Q^{\delta *} \neq \phi$, we have $A \neq \phi$. Again, $A^{\delta *} = (U \cap Q \cap Q^{\delta *})^{\delta *} \supseteq U \cap Q^{\delta *} \supseteq U \cap Q^{\delta *} \cap Q = A$ i.e. $A \subseteq A^{\delta *}$. For each $a \in A$, define $f_a : X \to X$ given by $f_a(x) = a^{-1}x$, and $\mathscr{F} = \{f_a : a \in A\}$. Since $A \neq \phi$, $\mathscr{F} \neq \phi$ and each f_a is a homeomorphism. Let $G = V \cap \delta Int((P)^{\delta *}) \cap \Gamma_{\delta}(P)$. Now it is sufficient to show that $G \cap \mathscr{F}^{-1}(y) \notin \mathcal{I}$ for every $y \in \mathscr{F}(G)$. Because then by Theorem 14, $\mathscr{F}(G \cap P) = \mathscr{F}(B) = A^{-1}B$ is a non-empty δ-open set in X contained in $Q^{-1}P$. Let $y \in \mathscr{F}(G)$. Then $y = a^{-1}x$ for some $a \in A$ and $x \in G \Rightarrow \mathscr{F}^{-1}(y) = Aa^{-1}x$. Thus $x \in Aa^{-1}x \subseteq A^{\delta *}a^{-1}x$ (as $A \subseteq A^{\delta *}$) $\subseteq (Aa^{-1}x)^{\delta *}$ (by Lemma 3) $= (\mathscr{F}^{-1}(y))^{\delta *} \Rightarrow N_x \cap \mathscr{F}^{-1}(y) \notin \mathcal{I}$ for some $N_x \in \tau^{\delta}(x)$. So in particular, as (2) is similar to (1). □

Corollary 1. *Let $(X, \tau, .)$ be a topological group and $\mathcal{I}$ be an ideal on X such that $\tau \sim^{\delta} \mathcal{I}$. Let $P \in \mathcal{U}(X, \tau, \mathcal{I})$ and $Q \in \mathcal{P}(X) - \mathcal{I}$.*

1. *If $\mathcal{I}$ is right translation invariant, then $[Q \cap Q^{\delta *}]^{-1}[P \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P)]$ is a non-empty δ-open set contained in $Q^{-1}P$.*

2. *If $\mathcal{I}$ is left translation invariant, then $[P \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P)][Q \cap Q^{\delta *}]^{-1}$ is a non-empty δ-open set contained in PQ^{-1}.*

Proof. We only show that $Q^{\delta *} \neq \phi$ and $P \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P) \neq \phi$, the rest follows from Theorem 15 by taking $U = V = X$. In fact, if $Q^{\delta *} = \phi$, then $Q \cap Q^{\delta *} = \phi$ which gives in view of Theorem 3, $Q \in \mathcal{I}$, a contradiction.
Again, $P \in \mathcal{U}(X, \tau, \mathcal{I}) \Rightarrow \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P) \neq \phi$ (by Lemma 2 and Proposition 2) $\Rightarrow \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P) \in \tau^{\delta} - \phi$. Now, $\delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P) = [P \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P)] \cup [P^c \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P)] \notin \mathcal{I}$ (by Lemma 2). Then $[P^c \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P)] \subseteq [P^c \cap \Gamma_{\delta}(P)] = \Gamma_{\delta}(P) - P \in \mathcal{I}$ by Theorem 8. Thus $P \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P) \notin \mathcal{I}$ and hence $P \cap \delta Int(P^{\delta *}) \cap \Gamma_{\delta}(P) \neq \phi$. □

Corollary 2. *Let $(X, \tau, .)$ be a topological group and $\mathcal{I}$ be an ideal on X such that $\mathcal{I} \cap \tau^{\delta} = \phi$ and $P \in \mathcal{U}(X, \tau, \mathcal{I})$.*

1. *If $\mathcal{I}$ is left translation invariant, then $e \in \delta Int(P^{-1}P)$.*
2. *If $\mathcal{I}$ is right translation invariant, then $e \in \delta Int(PP^{-1})$.*
3. *If $\mathcal{I}$ is left as well as right translation invariant, then $e \in \delta Int(PP^{-1} \cap P^{-1}P)$.*

Proof. It suffices to prove (1) only. We have, $P \in \mathcal{U}(X, \tau, \mathcal{I})$ then there exist $Q \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $Q \subseteq P$. Now for any $x \in X$, $\Gamma_\delta(Q)x \cap \Gamma_\delta(Q) = \Gamma_\delta(Qx) \cap \Gamma_\delta(Q) = \Gamma_\delta(Qx \cap Q)$ (by Lemma 3 and Theorem 6). Thus if $\Gamma_\delta(Q)x \cap \Gamma_\delta(Q) \neq \phi$, then $Qx \cap Q \neq \phi$. Now, if $x \in [\Gamma_\delta(Q)]^{-1}[\Gamma_\delta(Q)]$ then $x = y^{-1}z$ for some $y, z \in \Gamma_\delta(Q)$, then $yx = z = t$ (say) $\Rightarrow t \in \Gamma_\delta(Q)x$ and $t \in \Gamma_\delta(Q) \Rightarrow \Gamma_\delta(Q)x \cap \Gamma_\delta(Q) \neq \phi \Rightarrow x \in \{x \in X : \Gamma_\delta(Q)x \cap \Gamma_\delta(Q) \neq \phi\}$ then $[\Gamma_\delta(Q)]^{-1}[\Gamma_\delta(Q)] \subseteq \{x \in X : \Gamma_\delta(Q)x \cap \Gamma_\delta(Q) \neq \phi\} \subseteq \{x \in X : Qx \cap Q \neq \phi\} \subseteq Q^{-1}Q \subseteq P^{-1}P$. Since $\Gamma_\delta(Q) \neq \phi$ by Proposition 2 as $Q \in \mathcal{U}(X, \tau, \mathcal{I})$ and $\Gamma_\delta(Q)$ is δ-open for any $Q \subseteq X$, we have $e \in [\Gamma_\delta(Q)]^{-1}[\Gamma_\delta(Q) \subseteq \delta Int(P^{-1}P)$. □

References

[1] A. Al-Omari and E. Hatir, On Γ_δ-operator in ideal topological spaces (submitted).

[2] D. Janković and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97(1990), 295-310.

[3] T. R. Hamlett and D. Rose, Remarks on some theorems of banach, mcshans, and pettis, Rocky Mountain J. of Math., 22 (4) (1992), 1329-1339.

[4] T. R. Hamlett and D. Jankovic, Ideals in topological spaces and the set operator Ψ, Boll. Un. Mat. Ital. (7) 4-B (1990), 863-874.

[5] E. Hatir, A. Al-Omari and S. Jafari, δ-local function and its properties in ideal topological spaces (submitted).

[6] E. Hayashi, topologies defined by local properties, Math. ann., 156(1964), 205-215.

[7] K. Kuratowski, Topology I, Warszawa, 1933,

[8] M. Khan and T. Noiri, Semi-local functions in ideal topological spaces, J. of Advanced Research in Pure Math., 2(2010), 36-42.

[9] N. Levine, semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963), 36-41.

[10] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph. D. Dissertation, Univ. of Cal. at Santa Barbara, 1967.

[11] T. Noiri, On δ-continuous functions, J. Korean Math. Soc., 16 (2) 1990, 161-166.

[12] N. V. Veličko, H-closed topological spaces, Amer. Math. soc. Transl., 2(78)(1968), 103-118.

[13] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, 1960.

[14] S. Yuksel, A. Acikgoz and T. Noiri, On δ-$\mathcal{I}$-continuous functions, Turkish J. of Math., 29(2005), 39-51.

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