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TRIPLED COINCIDENCE POINT THEOREMS FOR A CLASS OF CONTRACTIONS IN ORDERED METRIC SPACES

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Abstract. In [Coupled fixed points for a class of contractions in partially ordered spaces and applications, J. Comput. Anal. Appl. vol. 13, no. 6, (2011), 1123-1131], Jiandong Yin introduced the concept of a mixed g-comparable mapping $F : X^2 \to X$ and proved coupled coincidence point theorems for a class of nonlinear contractive mappings. In this paper, we introduce the concept of a mixed g-comparable mapping $F : X^3 \to X$ and we present some tripled coincidence fixed point theorems for nonlinear contractions in the setting of partially ordered metric spaces.

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1. Introduction and preliminaries

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first important result on fixed points for contractive type mappings was the much celebrated Banach's contraction principle by Banach [9] in 1922. Because of its importance and usefulness for mathematical theory, its has been become a very popular tool in solving existence problems in many branches of mathematical analysis and it has been extended in many directions. Several authors have obtained various extensions and generalizations of Banach's theorem by considering contractive mappings on many different metric spaces. In [4], Alber and Guerre-Delabriere introduced the concept of weak contraction in Hilbert spaces. Rhoades [22] has shown that the result which Alber et al. proved in [4] is also valid in complete metric spaces.

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [21], and then by Nieto and López [20]. Further results in this direction under weak contraction condition were proved, see [2,3,4,5,10,11,12,13,16,17,19,25]. Various results on coupled fixed point have been obtained, for more details see [6,7,14,15,18,23].

Samet and Vetro [24] introduced the notion of fixed point of N-order as natural extension of that of coupled fixed point and established some new coupled fixed point theorems in complete metric spaces, using a new concept of F-invariant set. Recently, in the same spirit, the case $N = 3$ (tripled case) is treated by Berinde and Borcut [8]. We recall these known definitions:

Definition 1. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to has the mixed monotone property if for any $x, y, z \in X$*

$$x_1,\ x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1,\ y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1,\ z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 2.(see [8]) *Let $F : X \times X \times X \to X$. An element (x, y, z) is called a tripled fixed point of F if*

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z.$$

Berinde and Borcut [8] proved the following theorem:

Theorem 1. *Let $(X, \leq, d)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \to X$ such that F has the mixed monotone property and*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{1}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*
3. *if a non-decreasing sequence $z_n \to z$, then $z_n \leq z$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z,$$

that is, F has a tripled fixed point.

In this paper, we establish tripled coincidence point theorems for $F : X \times X \times X \to X$ and $g : X \to X$ satisfying a class of contractions in partially ordered metric spaces.

2. Main results

In a recent paper, Abbas, Aydi and Karapınar [1] introduced the following concepts:

Definition 3.(see [1]) *Let $(X, \leq)$ be a partially ordered set. Let $F : X \times X \times X \to X$ and $g : X \to X$. The mapping F is said to has the mixed g-monotone property if for any $x, y, z \in X$*

$$x_1,\ x_2 \in X, \quad gx_1 \leq gx_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1,\ y_2 \in X, \quad gy_1 \leq gy_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1,\ z_2 \in X, \quad gz_1 \leq gz_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 4.(see [1]) *Let $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled coincidence point of F and g if*

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad F(z, y, x) = gz.$$

(gx, gy, gz) is said a tripled point of coincidence of F and g.

Definition 5.(see [1]) *Let $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled common fixed point of F and g if*

$$F(x, y, z) = gx = x, \quad F(y, x, y) = gy = y, \quad F(z, y, x) = gz = z.$$

Definition 6.*Let X be a non-empty set. Then we say that the mappings $F : X \times X \times X \to X$ and $g : X \to X$ are commutative if for all $x, y, z \in X$*

$$g(F(x, y, z)) = F(gx, gy, gz).$$

Yin [26] introduced the following concepts:

Definition 7.(see [26]) *Let $(X, \leq)$ be a partially ordered set and $x, y \in X$. The pair (x, y) is called comparable if either $x \leq y$ or $y \leq x$ holds. We say that the pairs (a, b) and (x, y) are comparable in the same direction if either $x \leq y$ and $a \leq b$ or $y \leq x$ and $b \leq a$ hold. We say that the pairs (a, b) and (x, y) are comparable in the opposite direction if either $x \leq y$ and $a \geq b$ or $y \leq x$ and $b \geq a$ hold. A sequence $\{x_n\} \subset X$ is comparable if x_n and x_{n+1} are comparable for each $n = 0, 1, 2, \cdots$.*

Definition 8.(see [26]) *Let $(X, \leq)$ be a partially ordered set. We say that F has the mixed g-comparable property if for any $x_1,\ x_2,\ y_1,\ y_2 \in X$, if $g(x_1),\ g(x_2)$ are comparable and $g(y_1),\ g(y_2)$ are comparable implies that $F(x_1, y_1),\ F(x_2, y_2)$ are comparable.*

Inspired with Definition 8, we introduce the following concept of a mixed g-comparable mapping $F : X^3 \to X$.

Definition 9.*Let $(X, \leq)$ be a partially ordered set. A mapping $F : X^3 \to X$ is said to have a mixed g-comparable property if for any $(x, y, z), (p, r, s) \in X^3$ we have the* *following property: if the pairs* $(g(x), g(p)), (g(y), g(r)))$ *and* $(g(z), g(s))$ *are comparable then* $F(x, y, z)$ *and* $F(p, r, s)$ *are comparable.*

For a metric space (X, d), the function $\rho : X^3 \to [0, \infty)$, given by,

$$\rho((x, y, z), (u, v, r)) := d(x, u) + d(y, v) + d(z, r)$$

forms a metric space on X^3, that is, (X^3, ρ) is a metric induced by (X, d).

Let Φ denote all the functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying $\varphi(t) < t$ for each $t > 0$. Our result is the following:

Theorem 2. *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X*. Suppose* $F : X \times X \times X \to X$ *and* $g : X \to X$ *are such that* F *has the mixed* g*-comparable property and Assume also that there exist* $p, q, r \in [0, 1)$ *with* $p + 2q + r < 1$ *such that*

$$d(F(x, y, z), F(u, v, w)) \leq \varphi\Bigg(pd(gx, gu) + qd(gy, gv) + rd(gz, gw)\Bigg), \tag{2}$$

for all $x, y, z, u, v, w \in X$ *for which the pairs* $(g(x), g(u)), (g(y), g(v)))$ *and* $(g(z), g(w))$ *are comparable, where* $\varphi \in \Phi$*. Suppose* $F(X \times X \times X) \subset g(X)$ *and* $g(X)$ *is a complete subset of* (X, d)*. Suppose also* X *has the following properties:*

- (i) *if a comparable sequence* $x_n \to x$*, then the pairs* (x_n, x) *are comparable for all* n*,*
- (ii) *if a comparable sequence* $y_n \to x$ *, then the pairs* (y_n, y) *are comparable for all* n*,*
- (iii) *if a comparable sequence* $z_n \to x$ *, then the pairs* (z_n, z) *are comparable for all* n*,*
- (iv) *if* $\{t_n\}$ *is comparable sequence, then for any* n *and* m*, the pair* (t_n, t_m) *is comparable.*

If there exist $x_0, y_0, z_0 \in X$ *such that the pairs* $(gx_0, F(x_0, y_0, z_0)), (gy_0, F(y_0, x_0, y_0))$ *and* $(gz_0, F(z_0, y_0, x_0))$ *are comparable, then there exist* $x, y, z \in X$ *such that*

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad F(z, y, x) = gz,$$

that is, F *and* g *have a tripled coincidence point.*

Proof. Let $x_0, y_0, z_0 \in X$ such that the pairs $(gx_0, F(x_0, y_0, z_0)), (gy_0, F(y_0, x_0, y_0))$ and $(gz_0, F(z_0, y_0, x_0))$ are comparable. Since $F(X \times X \times X) \subset g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that

$$gx_1 = F(x_0, y_0, z_0), \quad gy_1 = F(y_0, x_0, y_0) \quad \text{and} \quad gz_1 = F(z_0, y_0, x_0). \tag{3}$$

Again, from $F(X \times X \times X) \subset g(X)$, continuing this process, we can construct sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, x_n, z_n) \quad \text{and} \quad gz_{n+1} = F(z_n, y_n, x_n). \tag{4}$$

We shall show that

$$(gx_n, gx_{n+1}), \quad (gy_n, gy_{n+1}) \text{ and } (gz_n, gz_{n+1}) \text{ are comparable for all } n \geq 0, \tag{5}$$

We shall use the mathematical induction. Due to the assumption we know that the pairs $(gx_0, F(x_0, y_0, z_0)), (gy_0, F(y_0, x_0, y_0))$ and $(gz_0, F(z_0, y_0, x_0))$ are comparable. As $gx_1 = F(x_0, y_0, z_0), gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$, then $(gx_0, gx_1), (gy_0, gy_1)$ and (gz_0, gz_1) are comparable for all $n \geq 0$. Thus, the assertion (5) holds for $n = 0$.

Suppose (5) holds for some $n \geq 0$. Regarding (5) and by mixed g-comparable property of F and (4), we have

$$gx_{n+1} = F(x_n, y_n, z_n) \text{ is comparable to } gx_{n+2} = F(x_{n+1}, y_{n+1}, z_{n+1}). \tag{6}$$

Analogously, we can get that the pairs $(gz_n, gz_{n+1}), (gy_n, gy_{n+1})$ are comparable. Thus, (5) holds for any $n \in \mathbb{N}$, that is, $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are comparable sequences.

If for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \quad gy_n = gy_{n+1} \text{ and } \quad gz_n = gz_{n+1},$$

then, by (4), (x_n, y_n, z_n) is a tripled coincidence point of F and g. From now on, assume for any $n \in \mathbb{N}$ that at least

$$gx_n \neq gx_{n+1} \quad \text{or} \quad gy_n \neq gy_{n+1} \quad \text{or} \quad gz_n \neq gz_{n+1}. \tag{7}$$

Set

$$\delta_{n+1} = \rho((gx_n, gy_n, gz_n), (gx_{n+1}, gy_{n+1}, gz_{n+1})) = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}).$$

Due to assumption (iv) of theorem, and (2), (4), we have

$$d(gx_n, gx_{n+1}) \leq \varphi(pd(gx_{n-1}, gx_n) + qd(gy_{n-1}, gy_n) + rd(gz_{n-1}, gz_n)), \tag{8}$$

$$d(gy_n, gy_{n+1}) \leq \varphi(pd(gy_{n-1}, gy_n) + qd(gx_{n-1}, gx_n) + rd(gy_{n-1}, gy_n)), \tag{9}$$

and

$$d(gz_n, gz_{n+1}) \leq \varphi(pd(gz_{n-1}, gz_n) + qd(gy_{n-1}, gy_n) + rd(gx_{n-1}, gx_n)). \tag{10}$$

Thus, from (8)-(10) and using the property $\varphi(t) < t$, we obtain that

$$\begin{aligned} & d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) \\ \leq & [p+q+r]d(gx_{n-1}, gx_n) + [p+2q+r]d(gy_{n-1}, gy_n) + [p+r]d(gz_{n-1}, gz_n) \\ \leq & [p+2q+r]\left(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)\right) \end{aligned} \tag{11}$$

Take $k = p + 2q + r$, then (11) becomes

$$\delta_{n+1} \leq k\delta_n, \tag{12}$$

which implies that

$$\delta_n \leq k\delta_{n-1} \leq k^{n-1}\delta_1. \tag{13}$$

By assumption, we have $0 \leq p + 2q + r < 1$, then $k \in [0, 1)$. ¿From (13) we get that

$$\lim_{n\to\infty} \delta_n = 0, \tag{14}$$

that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})] = 0. \tag{15}$$

Now, we shall prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ is not Cauchy. So, there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}$; $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$ and $\{gz_{n(k)}\}$, $\{gz_{m(k)}\}$ of $\{gz_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)}) \geq \varepsilon. \tag{16}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (16) and $n(k) > m(k) \geq k$. Thus,

$$d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)-1}, gy_{m(k)}) + d(gz_{n(k)-1}, gz_{m(k)}) < \varepsilon. \tag{17}$$

By using triangle inequality and having (16), (17) in mind

$$\begin{aligned} \varepsilon \ & \leq t_k = d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)}) \\ & \leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)}) \\ & \quad + d(gz_{n(k)}, gz_{n(k)-1}) + d(gz_{n(k)-1}, gz_{m(k)}) \\ & < d(gx_{n(k)}, gx_{n(k)-1}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gz_{n(k)}, gz_{n(k)-1}) + \varepsilon. \end{aligned} \tag{18}$$

Letting $k \to \infty$ in (18) and using (15)

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)}) = \varepsilon. \tag{19}$$

Again by triangle inequality,

$$\begin{aligned} t_k &= d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)}) \\ &\leq d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)}) \\ &\quad + d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)}) \\ &\quad + d(gz_{n(k)}, gz_{n(k)+1}) + d(gz_{n(k)+1}, gz_{m(k)+1}) + d(gz_{m(k)+1}, gz_{m(k)}) \\ &\leq \delta_{n(k)+1} + \delta_{m(k)+1} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) \\ &\quad + d(gz_{n(k)+1}, gz_{m(k)+1}). \end{aligned} \tag{20}$$

We have in mind

$$(gx_{n(k)}, gx_{m(k)}),\ \ (gy_{n(k)}, gy_{m(k)}) \text{ and } (gz_{n(k)}, gz_{m(k)}) \text{ are comparable.} \tag{21}$$

Hence from (2), (4) and (21), we have using $\varphi(t) < t$

$$\begin{aligned} d(gx_{n(k)+1}, gx_{m(k)+1}) &= d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}) \\ &\leq \varphi\left(pd(gx_{n(k)}, gx_{m(k)}) + qd(gy_{n(k)}, gy_{m(k)}) + rd(gz_{n(k)}, gz_{m(k)})\right) \\ &< pd(gx_{n(k)}, gx_{m(k)}) + qd(gy_{n(k)}, gy_{m(k)}) + rd(gz_{n(k)}, gz_{m(k)}) \end{aligned} \tag{22}$$

$$\begin{aligned} d(gy_{n(k)+1}, gy_{m(k)+1}) &= d(F(y_{n(k)}, x_{n(k)}, y_{n(k)}), F(y_{m(k)}, x_{m(k)}, y_{m(k)})) \\ &\leq \varphi\left(p(gy_{n(k)}, gy_{m(k)}) + qd(gx_{n(k)}, gx_{m(k)}) + rd(gy_{n(k)}, gy_{m(k)})\right) \\ &< pd(gy_{n(k)}, gy_{m(k)}) + qd(gx_{n(k)}, gx_{m(k)}) + rd(gy_{n(k)}, gy_{m(k)}) \end{aligned} \tag{23}$$

$$\begin{aligned} d(gz_{n(k)+1}, gz_{m(k)+1}) &= d(F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{m(k)}, y_{m(k)}, x_{m(k)})) \\ &\leq \varphi\left(pd(gz_{n(k)}, gz_{m(k)}) + qd(gy_{n(k)}, gy_{m(k)}) + qd(gx_{n(k)}, gx_{m(k)})\right) \\ &< pd(gz_{n(k)}, gz_{m(k)}) + qd(gy_{n(k)}, gy_{m(k)}) + qd(gx_{n(k)}, gx_{m(k)}). \end{aligned} \tag{24}$$

¿From (22)-(24) we get that

$$\begin{aligned} &d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) \\ \leq&(p+q+r)d(gx_{n(k)}, gx_{m(k)}) + (p+2q+r)d(gy_{n(k)}, gy_{m(k)}) + (p+r)d(gz_{n(k)}, gz_{m(k)}) \\ \leq&(p+2q+r)\left(d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)})\right) \\ =&(p+2q+r)t_k. \end{aligned} \tag{25}$$

Combining (20) with (25), we obtain that

$$\begin{aligned} t_k &\leq \delta_{n(k)+1} + \delta_{m(k)+1} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) \\ &\quad + d(gz_{n(k)+1}, gz_{m(k)+1}) \\ &\leq \delta_{n(k)+1} + \delta_{m(k)+1} + (p+2q+r)t_k. \end{aligned} \tag{26}$$

Letting $k \to \infty$ and having in mind (15) we get

$$\varepsilon \leq (p + 2q + r)\varepsilon < \varepsilon$$

which is a contradiction. This shows that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences.

Since $g(X)$ is complete, there exist $x, y, z \in X$ such that

$$\lim_{n\to+\infty} gx_n = gx, \quad \lim_{n\to+\infty} gy_n = gy, \quad \text{and} \quad \lim_{n\to+\infty} gz_n = gz. \tag{27}$$

Due to properties $(i) - (iii)$, for any $n \geq 0$, the pairs $(gx_n, gx), (gy_n, gy)$ and (gz_n, gz) are comparable. Hence, by (2) we have

$$d(F(x_n, y_n, z_n), F(x, y, z)) \leq \varphi\left(pd(gx_n, gx) + qd(gy_n, gy) + rd(gz_n, gz)\right). \tag{28}$$

Since $\varphi(t) < t$ for each $t > 0$, then we have $\lim_{r\to 0^+} \varphi(r) = 0$. Letting $n \to \infty$ in (28), we get that $\lim_{n\to\infty} d(F(x_n, y_n, z_n), F(x, y, z)) = 0$. Thus, $\lim_{n\to\infty} F(x_n, y_n, z_n) = F(x, y, z)$ and by (27)

$$gx = \lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} F(x_n, y_n, z_n) = F(x, y, z).$$

Analogously we have $F(y, x, y) = gy$ and $F(z, y, x) = gz$. Hence F and g have a tripled coincidence point.

Corollary 1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X. Suppose $F : X \times X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-comparable property and assume also that there exist $p, q, r, k \in [0, 1)$ with $p + 2q + r < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq kpd(gx, gu) + kqd(gy, gv) + krd(gz, gw), \tag{29}$$

for all $x, y, z, u, v, w \in X$ for which the pairs $(g(x), g(u)), (g(y), g(v)))$ and $(g(z), g(w))$ are comparable. Suppose $F(X \times X \times X) \subset g(X)$ and $g(X)$ is a complete subset of (X, d). Suppose also X has the following properties:

- (i) *if a comparable sequence $x_n \to x$, then the pairs (x_n, x) are comparable for all n,*
- (ii) *if a comparable sequence $y_n \to x$, then the pairs (y_n, y) are comparable for all n,*
- (iii) *if a comparable sequence $z_n \to x$, then the pairs (z_n, z) are comparable for all n,*

(iv) *if* $\{t_n\}$ *is comparable sequence, then for any* n *and* m*, the pair* (t_n, t_m) *is comparable.*

If there exist $x_0, y_0, z_0 \in X$ *such that the pairs* $(gx_0, F(x_0, y_0, z_0)), (gy_0, F(y_0, x_0, y_0))$ *and* $(gz_0, F(z_0, y_0, x_0))$ *are comparable, then there exist* $x, y, z \in X$ *such that*

$$F(x,y,z) = gx, \quad F(y,x,y) = gy, \quad F(z,y,x) = gz,$$

that is, F *and* g *have a tripled coincidence point.*

Proof. Taking $\varphi(t) = kt$ in Theorem 2 we obtain Corollary 1.

Recall that $(X, \leq)$ is a partially ordered set and d is a metric on X such that (X, d) is a complete metric space. Further, we endow the product space X^3 with the following partial order:

$$\text{for } \ (x,y,z),\ (a,b,c) \in X^3,\ (x,y,z) \leq (a,b,c) \Longleftrightarrow x \leq a,\ y \geq b \text{ and } z \leq c.$$

We say that (x, y, z) is T-comparable to (a, b, c) if $(x, y, z) \leq (a, b, c)$ or $(a, b, c) \leq (x, y, z)$, that is, the pair (x, a) is comparable in the same direction to (z, c) and comparable to (y, b) in opposite direction.

Theorem 3.*In addition to hypothesis of Theorem 2, suppose that for all* $(x, y, z), (u, v, r) \in X \times X \times X$*, there exists* $(a, b, c) \in X \times X \times X$ *such that* $(F(a, b, c), F(b, a, b), F(c, b, a))$ *is* T*-comparable to* $(F(x, y, z), F(y, x, y), F(z, y, x))$ *and* $(F(u, v, r), F(v, u, v), F(r, v, u))$*. Also, assume that* F *commutes with* g*. Then,* F *and* g *have a unique tripled common fixed point* (x, y, z) *such that*

$$x = gx = F(x,y,z), \quad y = gy = F(y,x,y) \quad \text{and} \quad z = gz = F(z,y,x).$$

Proof. The set of tripled coincidence points of F and g is not empty due to Theorem 2. Assume, now, (x, y, z) and (u, v, r) are two tripled coincidence points of F and g, that is,

$$\begin{array}{ll} F(x,y,z) = gx, & F(u,v,r) = gu, \\ F(y,x,y) = gy, & F(v,u,v) = gv, \\ F(z,y,x) = gz, & F(r,v,u) = gr, \end{array}$$

We shall show that (gx, gy, gz) and (gu, gv, gr) are equal. By assumption, there exists $(a, b, c) \in X \times X \times X \times X$ such that $(F(a, b, c), F(b, a, b), F(c, b, a))$ is T-comparable to $(F(x, y, z), F(y, x, y), F(z, y, x))$ and $(F(u, v, r), F(v, u, v), F(r, v, u))$.

Define sequences $\{ga_n\}, \{gb_n\}$ and $\{gc_n\}$ such that

$$a = a_0, \quad b = b_0, \quad c = c_0, \quad \text{and}$$

$$\begin{aligned} ga_n &= F(a_{n-1}, b_{n-1}, c_{n-1}),\\ gb_n &= F(b_{n-1}, a_{n-1}, b_{n-1}),\\ gc_n &= F(c_{n-1}, b_{n-1}, a_{n-1}), \end{aligned} \tag{30}$$

for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$ and $u_0 = u$, $v_0 = v$, $r_0 = r$, and on the same way define the sequences $\{gx_n\}, \{gy_n\}$, $\{gz_n\}$ and $\{gu_n\}, \{gv_n\}$, $\{gr_n\}$. Then, it is easy that

$$\begin{aligned} gx_n &= F(x, y, z), \qquad & gu_n &= F(u, v, r),\\ gy_n &= F(y, x, y,), \qquad & gv_n &= F(v, u, v),\\ gz_n &= F(z, y, x), \qquad & gr_n &= F(r, v, u), \end{aligned} \tag{31}$$

for all $n \geq 1$. Since $(F(x, y, z), F(y, x, y), F(z, y, x)) = (gx_1, gy_1, gz_1) = (gx, gy, gz)$ is T-comparable to $(F(a, b, c), F(b, a, b), F(c, b, a)) = (ga_1, gb_1, gc_1)$, therefore $(gx, gy, gz) \geq (ga_1, gb_1, gc_1)$. Recursively, we have

$$(gx, gy, gz) \geq (ga_n, gb_n, gc_n) \quad \text{for all } n. \tag{32}$$

This implies that (gx, ga_n), (gy, gb_n) and (gz, gc_n) are comparable. By this and (2), we have

$$\begin{aligned} d(gx, ga_{n+1}) &= d(F(x, y, z), F(a_n, b_n, c_n))\\ &\leq \varphi(pd(gx, ga_n) + qd(gy, gb_n) + rd(gz, gc_n)), \end{aligned} \tag{33}$$

$$\begin{aligned} d(gb_{n+1}, gy) &= d(F(b_n, a_n, b_n), F(y, x, y))\\ &\leq \varphi(pd(gb_n, gy) + qd(ga_n, gx) + rd(gb_n, gy)), \end{aligned} \tag{34}$$

and

$$\begin{aligned} d(gz, gc_{n+1}) &= d(F(z, y, x), F(c_n, b_n, a_n)\\ &\leq \varphi(pd(gz, gc_n) + qd(gy, gb_n) + rd(gx, ga_n)). \end{aligned} \tag{35}$$

Set

$$\gamma_n = d(gx, ga_n) + d(gy, gb_n) + d(gz, gc_n).$$

We deduce from (33)-(35) and the property $\varphi(t) < t$, that

$$\gamma_{n+1} \leq k\, \gamma_n,$$

where $k = p + 2q + r < 1$. It follows that

$$\gamma_n \leq k^n \gamma_0. \tag{36}$$

Therefore, $\lim\limits_{n\to+\infty} \gamma_n = 0$. Thus,

$$\begin{aligned} &\lim_{n\to\infty} d(gx, ga_n) = 0, \quad \lim_{n\to\infty} d(gy, gb_n) = 0,\\ &\lim_{n\to\infty} d(gz, gc_n) = 0. \end{aligned} \tag{37}$$

Analogously, we show that

$$\begin{aligned}\lim_{n\to\infty} d(gu, ga_n) = 0, \quad \lim_{n\to\infty} d(gv, gb_n) = 0,\\ \lim_{n\to\infty} d(gr, gc_n) = 0.\end{aligned} \tag{38}$$

Combining (37) and (38) yields that (gx, gy, gz) and (gu, gv, gr) are equal. Since $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, x)$, by commutativity of F and g, we have

$$gx' = g(gx) = g(F(x, y, z)) = F(gx, gy, gz),$$

$$gy' = g(gy) = g(F(y, x, y)) = F(gy, gx, gy),$$

and

$$gz' = g(gz) = g(F(z, y, x)) = F(gz, gy, gx),$$

where $gx = x'$, $gy = y'$ and $gz = z'$. Thus, (x', y', z') is a tripled coincidence point of F and g. Consequently, (gx', gy', gz') and (gx, gy, gz) are equal. We deduce

$$gx' = gx = x', \quad gy' = gy = y' \quad \text{and} \quad gz' = gz = z'.$$

Therefore, (x', y', z') is a tripled common fixed of F and g. Its uniqueness follows easily from (2).

Example 1. *Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the usual ordering.*
Set $gx = x$. Let $F : X^3 \to X$ be given by

$$F(x, y, z) = \frac{6x - 6y + 6z + 5}{36}, \text{ for all } x, y, z \in X$$

Let $\varphi : [0, \infty) \to [0, \infty)$ be given by $\varphi(t) = \frac{t}{2}$ for all $t \in [0, \infty)$.
Take $p = q = r = \frac{1}{6}$ and $x_0 = y_0 = z_0 = \frac{1}{6}$. It is easy to check that all the conditions of Theorem 2 are satisfied and $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ is the unique common tripled fixed point of F and g.

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