

DETERMINATION OF STURM–LIOUVILLE OPERATOR ON A THREE-STAR GRAPH FROM FOUR SPECTRA

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ABSTRACT. In this paper, we study determination of Sturm–Liouville operator on a three-star graph with the Dirichlet and Robin boundary conditions in the boundary vertices and matching conditions in the internal vertex from four spectra. We introduce an adequate Hilbert space formulation in such a way that the problem under consideration can be interpreted as an eigenvalue problem for a suitable self-adjoint operator. As spectral characteristics, we consider the spectrum of the main problem together with the spectra of two Dirichlet–Dirichlet problems and one Robin–Dirichlet problem on the edges of the graph and investigate their properties and asymptotic behavior. We prove that if these four spectra do not intersect, then the inverse problem of recovering the operator is uniquely solvable. We give an algorithm for the solution of the inverse problem with respect to this quadruple of spectra.

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1. INTRODUCTION

This paper is devoted to the study of the determination of Sturm–Liouville operators on a three-star graph with the Dirichlet and Robin boundary conditions in the boundary vertices and matching conditions in the internal vertex from four spectra. The considered inverse problem consists of recovering the Sturm–Liouville operator on a graph from the given spectral characteristics. Differential operators on graphs(networks, trees) often appear in mathematics, mechanics, physics, geophysics, physical chemistry, electronics, nanoscale technology and branches of natural sciences and engineering(see [2,6,10-12,19,28] and the bibliographies thereof). In recent years there has been considerable interest in the spectral theory of Sturm–Liouville operators on graphs(see [5,26,27]). The direct spectral and scattering problems on compact and noncompact graphs, respectively, were considered in many publications(see, for example [1,4,8,17,18]). The considered inverse spectral problem is not studied yet. However, inverse spectral problems of recovering differential

operators on star-type graphs with the boundary conditions other than considered here, were studied in [22,24] and other papers. Hochstadt-Lieberman type inverse problems on star-type graphs were investigated in [22,23].

We consider a three-star graph G with vertex set $V = \{v_0, v_1, v_2, v_3\}$ and edge set $E = \{e_1, e_2, e_3\}$, where v_1, v_2, v_3 are the boundary vertices, v_0 is the internal vertex and $e_j = [v_j, v_0]$ for $j = 1, 2, 3$. We assume that the length of every edge is equal to a , $a > 0$. Every edge $e_j \in E$ is viewed as an interval $[0, a]$. Parametrize $e_j \in E$ by $x \in [0, a]$, the following choice of orientation is convenient for us: $x = 0$ corresponds to the boundary vertices v_1, v_2, v_3 and $x = a$ corresponds to the internal vertex v_0 . A function Y on G may be represented as a vector $Y(x) = [y_j(x)]_{j=1,2,3}$, $x \in [0, a]$ and the function $y_j(x)$ is defined on the edge e_j . Let $q(x) = [q_j(x)]_{j=1,2,3}$ be a function on G which is called the potential and $q_j(x) \in L_2(0, a)$ is a real-valued function defined on the edge e_j . Let us consider the following Sturm–Liouville equations on G :

$$-y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), \quad x \in [0, a], \quad j = 1, 2, 3, \quad (1)$$

where λ is the spectral parameter. The functions $y_j(x)$ and $y_j'(x)$ are absolutely continuous and satisfy the following matching conditions in the internal vertex v_0 :

$$\left. \begin{aligned} y_i(a) &= y_j(a) \quad \text{for } i, j = 1, 2, 3, \quad (\text{continuity condition}), \\ \sum_{j=1}^3 y_j'(a) + \beta y_1(a, \lambda) &= 0 \quad (\text{Kirchhoff's condition}), \end{aligned} \right\} \quad (2)$$

where β is a real number. In electrical circuits, (2) expresses Kirchhof's law; in an elastic string network, it expresses the balance of tension and so on. Let us denote by L_0 the boundary-value problem for (1) with the matching conditions (2) and the following boundary conditions at the boundary vertices v_1, v_2, v_3 :

$$y_1(0) = y_2(0) = y_3'(0) - h y_3(0) = 0, \quad (3)$$

where h is a real number.

The problem of small transverse vibrations of a three-star graph consisting of three inhomogeneous smooth strings joined at the internal vertex with two pendent ends fixed and one pendent end can move without friction in the directions orthogonal to their respective equilibrium positions can be reduced to this problem by the Liouville transformation. This problem occurs also in quantum mechanics when one considers a quantum particle subject to the Shrödinger equation moving in a quasi-one-dimensional graph domain.

In this paper, we study the inverse problem of recovering the potential $q(x) = [q_j(x)]_{j=1,2,3}$ and the real numbers h and β from the given spectral characteristics.

Similar inverse spectral problems on star-type graphs with three and arbitrary number of edges but only with the Dirichlet conditions at the boundary vertices were considered in [23,24]. As spectral characteristics, we consider the set of eigenvalues of problem L_0 together with the sets of eigenvalues of the following two Dirichlet–Dirichlet problems and one Robin–Dirichlet problem on the edges of the graph G :

$$\begin{cases} -y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), & x \in [0, a], \\ y_j(0) = y_j(a) = 0, & j = 1, 2, \end{cases}$$

$$\begin{cases} -y_3''(x) + q_3(x)y_3(x) = \lambda^2 y_3(x), & x \in [0, a], \\ y_3'(0) - h y_3(0) = y_3(a) = 0, \end{cases}$$

which we denote these problems by L_j , $j = 1, 2, 3$. We obtain conditions for four sequences of real numbers that enable one to reconstruct the potential $q(x) = [q_j(x)]_{j=1,2,3}$ and the real numbers h and β so that one of the sequences describes the spectrum of the boundary-value problem L_0 and other three sequences coincide with the spectra of the problems L_j , $j = 1, 2, 3$. We give an algorithm for the construction of the potential and the coefficients of the boundary and matching conditions corresponding to these four sequences.

Denote by L'_j , $j = 1, 2, 3$ the following boundary-value problems:

$$\begin{cases} -y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), & x \in [0, a], \\ y_j(0) = y_j'(a) = 0, & j = 1, 2, \end{cases}$$

$$\begin{cases} -y_3''(x) + q_3(x)y_3(x) = \lambda^2 y_3(x), & x \in [0, a], \\ y_3'(0) - h y_3(0) = y_3'(a) = 0. \end{cases}$$

The main idea of the solution of the inverse problem for the considered system is its reduction to three independent inverse problems of reconstruction of the functions $q_j(x) \in L_2(0, a)$, $j = 1, 2, 3$ and h on the basis of two spectra, namely, the spectrum of the problem L_j and the spectrum of the problem L'_j . Since the solutions of the later inverse problems are known (see [7, Section 1.5], [16, Section 3.4]), this reduction gives an algorithm for the reconstruction of the potential and coefficients of the boundary-value problem L_0 .

This paper has the following structure: In section 2 we formulate the boundary value problem L_0 as an operator in an adequate Hilbert space. In Section 3 the direct problem is considered. Aspects of the theory of entire and meromorphic functions are used as tools for a description of the set of eigenvalues of the boundary-value problem L_0 and the spectra of the auxiliary problems L_j , $j = 1, 2, 3$ associated with this system. As a consequence we prove that the eigenvalues of the main problem and the spectra of the auxiliary problems interlace in some sense. In Section 4 we solve the inverse spectral problem for L_0 within the framework of the statement indicated above.

2. OPERATOR EQUATION FORMULATION

Let us consider the operator-theoretical interpretation of the problem L_0 . Denote by A the operator acting in the Hilbert space $H = L_2(0, a) \oplus L_2(0, a) \oplus L_2(0, a)$ with standard inner product $\langle \cdot, \cdot \rangle_H$, according to the formulas

$$AY = A \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \begin{pmatrix} -y_1''(x) + q_1(x)y_1(x) \\ -y_2''(x) + q_2(x)y_2(x) \\ -y_3''(x) + q_3(x)y_3(x) \end{pmatrix}, \tag{4}$$

$$D(A) = \left\{ \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} \left| \begin{array}{l} y_j(x) \in W_2^2(0, a) \text{ for } j = 1, 2, 3, \\ y_i(a) = y_j(a) \text{ for } i, j = 1, 2, 3, \\ \sum_{j=1}^3 y_j'(a) + \beta y_1(a) = 0, \\ y_1(0) = y_2(0) = y_3'(0) - hy_3(0) = 0 \end{array} \right. \right\}, \tag{5}$$

where $W_2^2(0, a)$ is a Sobolev space. It is clear that the squares of eigenvalues of the boundary value problem L_0 coincide with those of A .

Lemma 2.1. $D(A)$ is Dense in H .

Proof. Suppose that $F = (f_1(x), f_2(x), f_3(x))^t \in H$ is orthogonal to all $G = (g_1(x), g_2(x), g_3(x))^t \in D(A)$ (t denotes the transpose of a matrix), i.e.,

$$\langle F, G \rangle_H = \sum_{j=1}^3 \int_0^a f_j(x)g_j(x) = 0.$$

Since $C_0^\infty[0, a] \oplus 0 \oplus 0 \subseteq D(A)$ (Here 0 is a function that identically zero on $[0, a]$), then $G = (g_1(x), 0, 0) \in C_0^\infty[0, a] \oplus 0 \oplus 0$ is orthogonal to F , i.e.,

$$\langle F, G \rangle_H = \int_0^a f_1(x)g_1(x) = 0.$$

Since $C_0^\infty[0, a]$ is dense in $L_2(0, a)$, we must have $f_1(x) = 0$. Similarly, we get that $f_2(x) = f_3(x) = 0$. Thus, $D(A)$ is dense in H .

Theorem 2.2. The operator A is self-adjoint in the Hilbert space H .

Proof. Let $F = (f_1(x), f_2(x), f_3(x))^t$ and $G = (g_1(x), g_2(x), g_3(x))^t$ be arbitrary elements of $D(A)$. By twice integration by parts, we have

$$\langle AF, G \rangle_H = \langle F, AG \rangle_H + \sum_{j=1}^3 (f_j g_j' - f_j' g_j) \Big|_0^a.$$

It follows from (2) and (3) that $\sum_{j=1}^3 (f_j g'_j - f'_j g_j) \Big|_0^a = 0$. This yields,

$$\langle AF, G \rangle_H = \langle F, AG \rangle_H.$$

Therefore, A is symmetric in H . It remains to show that if $(AY, V)_H = (Y, U)_H$ for all $Y = (y_1(x), y_2(x), y_3(x))^t \in D(A)$, then $V \in D(A)$ and $AV = U$, where $V = (v_1(x), v_2(x), v_3(x))^t$ and $U = (u_1(x), u_2(x), u_3(x))^t$, i.e., (i) $v_j(x) \in W_2^2(0, a)$ ($j = 1, 2, 3$); (ii) $v_1(0) = v_2(0) = v'_3(0) - hv_3(0) = 0$; (iii) $v_j(a) = v_{j'}(a)$ ($j, j' = 1, 2, 3$); (iv) $\sum_{j=1}^3 v'_j(a) + \beta v_1(a) = 0$; (v) $\ell_j y_j = u_j$ ($j = 1, 2, 3$), where $\ell_j y_j := -y''_j + q_j y_j$.

For all $Y \in C_0^\infty(0, a) \oplus 0 \oplus 0 \subseteq D(A)$ (0 denotes the function identically zero on $[0, a]$), we have

$$\int_0^a (\ell_1 y_1) \bar{v}_1 dx = \int_0^a y_1 \bar{u}_1 dx.$$

So by standard Sturm–Liouville theory $v_1(x) \in W_2^2(0, a)$ and $u_1 = \ell_1 v_1$. Similarly we get $v_j(x) \in W_2^2(0, a)$ and $u_j = \ell_j v_j$ ($j = 2, 3$). Thus (i) and (v) hold. Now using (v) equation $(AY, V)_H = (Y, U)_H$ for all $Y \in D(A)$ becomes

$$\sum_{j=1}^3 \int_0^a (\ell_j y_j) \bar{v}_j dx = \sum_{j=1}^3 \int_0^a y_j \overline{\ell_j v_j} dx.$$

However by twice integration by parts, we have

$$\sum_{j=1}^3 \int_0^a (\ell_j y_j) \bar{v}_j dx = \sum_{j=1}^3 \int_0^a y_j \overline{\ell_j v_j} dx + \sum_{j=1}^3 (y_j \bar{v}'_j - y'_j \bar{v}_j) \Big|_0^a.$$

Hence

$$\sum_{j=1}^3 (y_j \bar{v}'_j - y'_j \bar{v}_j) \Big|_0^a = 0. \tag{6}$$

According to Naimark’s patching lemma (see [20, Part II, p. 63, Lemma 2]), there exists a $Y \in D(A)$ such that $y'_1(0) = 1, y_2(0) = y_3(0) = y'_3(0) = y_1(a) = y'_1(a) = y'_2(a) = y'_2(a) = y_3(a) = y'_3(a) = 0$. Then on account of equality (6), we have $v_1(0) = 0$. Similarly, we get $v_2(0) = v'_3(0) - hv_3(0) = 0$. So (ii) holds. Using Naimark’s patching lemma again one can show that (iii) and (iv) hold. consequently the operator A is self-adjoint.

Corollary 2.3. *The squares of eigenvalues of the boundary value problem L_0 are real.*

For all eigenvalues of the boundary-value problem L_0 to be real and nonzero, it is necessary and sufficient that the operator A be strictly positive ($A \gg 0$). Furthermore, integrating by parts, we obtain the following equality for any vector function $Y = (y_1(x), y_2(x), y_3(x))^t \in D(A)$ (t denotes the transpose of a matrix):

$$(AY, Y)_H = \sum_{j=1}^3 \int_0^a (|y_j'(x)|^2 + q_j(x)|y_j(x)|^2) dx + \beta|y_1(a)|^2 + h|y_3(0)|^2. \quad (7)$$

Relation (7) yields the following simple sufficient condition for the strict positivity of the operator A :

$$q_j(x) \geq \epsilon > 0 \quad \text{a.e. on } [0, a], \quad j = 1, 2, 3, \quad \beta \geq 0, \quad h \geq 0.$$

On the other hand, if $A \gg 0$, then setting in turn $Y = (y_1(x), 0, 0)^t \in D(A)$, $Y = (0, y_2(x), 0)^t \in D(A)$ and $Y = (0, 0, y_3(x))^t \in D(A)$ in (7), we establish that the eigenvalues of the problems L_j , $j = 1, 2, 3$ are also real and nonzero. The strict positivity of the operator A can be realized by shifting the spectral parameter $\lambda^2 - q_0$, $q_0 > 0$, in (1). For this reason, we assume in what follows without loss of generality that $A \gg 0$. Thus, the eigenvalues of the boundary-value problems L_0 and L_j , $j = 1, 2, 3$ are nonzero real numbers.

3. DIRECT PROBLEM

In this section, we describe the properties of sequences of eigenvalues of the boundary-value problems L_0 and L_j , $j = 1, 2, 3$ that are necessary for what follows.

Let us denote by $c_j(x, \lambda)$, $s_j(x, \lambda)$, $j = 1, 2, 3$ the solutions of (1) on the edge e_j which satisfy the initial conditions

$$c_j'(0, \lambda) = c_j(0, \lambda) - 1 = 0, \quad s_j(0, \lambda) = s_j'(0, \lambda) - 1 = 0. \quad (8)$$

For each fixed $x \in [0, a]$, the functions $c_j^{(\nu)}(x, \lambda)$ and $s_j^{(\nu)}(x, \lambda)$, $\nu = 0, 1$, $j = 1, 2, 3$ are entire in λ . Since $\{c_j(x, \lambda), s_j(x, \lambda)\}$ is a fundamental system of solutions of (1) on the edge e_j , then the solutions of (1) which satisfy the conditions (3), are

$$y_j(x, \lambda) = C_j u_j(x, \lambda), \quad j = 1, 2, 3, \quad (9)$$

where C_j , $j = 1, 2, 3$ are constants and

$$u_j(x, \lambda) = \begin{cases} s_j(x, \lambda), & j = 1, 2, \\ c_3(x, \lambda) + h s_3(x, \lambda), & j = 3. \end{cases} \quad (10)$$

Substituting (9) into (2), we establish that the eigenvalues of the boundary-value problem L_0 are zeros of the entire function

$$\Phi(\lambda) := \begin{vmatrix} u_1(a, \lambda) & -u_2(a, \lambda) & 0 \\ u_1(a, \lambda) & 0 & -u_3(a, \lambda) \\ u'_1(a, \lambda) + \beta u_1(a, \lambda) & u'_2(a, \lambda) & u'_3(a, \lambda) \end{vmatrix}$$

or

$$\Phi(\lambda) = \sum_{i=1}^3 \left(u'_i(a, \lambda) \prod_{\substack{j=1 \\ j \neq i}}^3 u_j(a, \lambda) \right) + \beta \prod_{j=1}^3 u_j(a, \lambda). \quad (11)$$

For what follows, we need the definition presented below:

Definition 3.1.([23]) Let $\{z_k\}_{-\infty}^{\infty}$ ($\{z_k\}_{-\infty, k \neq 0}^{\infty}$) be a sequence of complex numbers of finite multiplicities which satisfy the following conditions: (1) the sequence is symmetric with respect to the imaginary axis and symmetrically located numbers possess the same multiplicities; (2) any strip $|\operatorname{Re} z| \leq p < \infty$ contains not more than a finite number of z_k . Then, the following way of enumeration is called proper:

- i. $z_{-k} = -\overline{z_k}$ ($\operatorname{Re} z_k \neq 0$);
- ii. $\operatorname{Re} z_k \leq \operatorname{Re} z_{k+1}$;
- iii. the multiplicities are taken into account.

If a sequence has even number of pure imaginary elements we exclude the index zero from enumeration to make it proper.

Throughout section 3, denote

$$B_j = \begin{cases} \frac{1}{2} \int_0^a q_j(x) dx, & j = 1, 2, \\ h + \frac{1}{2} \int_0^a q_3(x) dx, & j = 3. \end{cases}$$

We introduce the entire function

$$\Psi(\lambda) = \prod_{j=1}^3 u_j(a, \lambda). \quad (12)$$

Let us denote by $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty}$ the set of zeros of $\Phi(\lambda)$ and by $\{\kappa_k\}_{-\infty, k \neq 0}^{\infty}$ the set of zeros of the function $\Psi(\lambda)$. Denote by $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^{\infty}$, $j = 1, 2, 3$ the sets of zeros

of the functions $u_j(a, \lambda)$, $j = 1, 2, 3$, respectively. It is clear from (12) that the set $\{\kappa_k\}_{-\infty, k \neq 0}^\infty$ is the union of the sets $\bigcup_{j=1}^3 \{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, i.e., the spectra of the auxiliary problems L_j , $j = 1, 2, 3$. According to the remark presented in Section 1, all numbers λ_k , $\nu_k^{(j)}$, $j = 1, 2, 3$ and κ_k are real and nonzero. We enumerate the sets $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$, $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, $j = 1, 2, 3$ and $\{\kappa_k\}_{-\infty, k \neq 0}^\infty$ in the proper way ($\lambda_{-k} = -\lambda_k$, $\lambda_k \leq \lambda_{k+1}$, $\nu_{-k}^{(j)} = -\nu_k^{(j)}$, $\nu_k^{(j)} < \nu_{k+1}^{(j)}$ for $j = 1, 2, 3$ and $\kappa_{-k} = -\kappa_k$, $\kappa_k \leq \kappa_{k+1}$). Note that the sets of eigenvalues $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, $j = 1, 2, 3$ behave asymptotically as follows (see [16, section 1.5]):

$$\nu_k^{(j)} = \frac{k\pi}{a} + \frac{B_j}{\pi k} + \frac{\delta_k^{(j)}}{k}, \quad j = 1, 2, \quad (13)$$

$$\nu_k^{(3)} = \frac{\pi(k - \frac{1}{2})}{a} + \frac{B_3}{\pi(k - \frac{1}{2})} + \frac{\delta_k^{(3)}}{k}, \quad (14)$$

where $\{\delta_k^{(j)}\}_{-\infty, k \neq 0}^\infty \in l_2$ for $j = 1, 2, 3$.

Let us denote by L^d , $d > 0$ the class (introduced in [13, p. 149]) of entire functions of exponential type $\leq d$ whose restrictions on the real line belong to $L_2(-\infty, \infty)$.

Lemma 3.2. *The functions $\Phi(\lambda)$ and $\Psi(\lambda)$ can be represented as follows:*

$$\begin{aligned} \Phi(\lambda) = & \frac{2 \sin \lambda a - 3 \sin^3 \lambda a}{\lambda} + (2B_1 + 2B_2 + 3B_3 + \beta) \frac{\sin^2 \lambda a \cos \lambda a}{\lambda^2} \\ & - (B_1 + B_2) \frac{\cos^3 \lambda a}{\lambda^2} + \frac{\omega_1(\lambda)}{\lambda^2}, \end{aligned} \quad (15)$$

$$\Psi(\lambda) = \frac{\sin^2 \lambda a \cos \lambda a}{\lambda^2} - (B_1 + B_2) \frac{\cos^2 \lambda a \sin \lambda a}{\lambda^3} + B_3 \frac{\sin^3 \lambda a}{\lambda^3} + \frac{\omega_2(\lambda)}{\lambda^3}, \quad (16)$$

where $\omega_1(\lambda), \omega_2(\lambda) \in L^{3a}$.

Proof. Using the formulas of [7, p. 18], [16, p. 9] and taking into account that

$$\int_0^a f(t) \cos \lambda t dt \in L^a, \quad \int_0^a f(t) \sin \lambda t dt \in L^a$$

whenever $f \in L_2(0, a)$ by the Paley-Wiener theorem [3, p. 103], we obtain

$$u_j(a, \lambda) = \frac{\sin \lambda a}{\lambda} + \frac{\varrho_{j1}(\lambda)}{\lambda} = \frac{\sin \lambda a}{\lambda} - B_j \frac{\cos \lambda a}{\lambda^2} + \frac{\varrho_{j2}(\lambda)}{\lambda^2}, \quad j = 1, 2, \quad (17)$$

$$u_3(a, \lambda) = \cos \lambda a + \varrho_{31}(\lambda) = \cos \lambda a + B_3 \frac{\sin \lambda a}{\lambda} + \frac{\varrho_{32}(\lambda)}{\lambda} \quad (18)$$

$$u'_j(a, \lambda) = \cos \lambda a + B_j \frac{\sin \lambda a}{\lambda} + \frac{\sigma_j(\lambda)}{\lambda} \quad j = 1, 2, \quad (19)$$

$$u'_3(a, \lambda) = -\lambda \sin \lambda a + B_3 \cos \lambda a + \sigma_3(\lambda), \quad (20)$$

where $\varrho_{j1}(\lambda)$, $\varrho_{j2}(\lambda)$, $\sigma_j(\lambda)$, $j = 1, 2, 3$, are entire functions of class L^a . Substituting (17)-(20) into (11) and (12), we get (15) and (16).

Theorem 3.3. *The set $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty}$ of zeros of $\Phi(\lambda)$ can be represented as the union of three pairwise disjoint subsequences $\bigcup_{j=1}^3 \{\lambda_k^{(j)}\}_{-\infty, k \neq 0}^{\infty}$ which being enumerated in the following way: $\lambda_{-k}^{(1)} = -\lambda_k^{(1)}$, $\lambda_{-k}^{(2)} = -\lambda_k^{(2)}$, $\lambda_{-k}^{(3)} = -\lambda_k^{(3)}$ and $\lambda_k^{(j)} \leq \lambda_{k+1}^{(j)}$ for $j = 1, 2, 3$, behave asymptotically as follows:*

$$\lambda_k^{(1)} = \frac{k\pi}{a} + \frac{B_1 + B_2}{2k\pi} + \frac{\gamma_k^{(1)}}{k}, \quad (21)$$

$$\lambda_k^{(j)} = \frac{k\pi + (-1)^j \sin^{-1} \sqrt{\frac{2}{3}}}{a} + \frac{3B_1 + 3B_2 + 6B_3 + 2\beta}{12k\pi} + \frac{\gamma_k^{(j)}}{k}, \quad j = 2, 3, \quad (22)$$

where $\{\gamma_k^{(j)}\}_{-\infty, k \neq 0}^{\infty} \in l_2$ for $j = 1, 2, 3$.

Proof. In the same way as [22, Lemma 1.3], we can show that the set of zeros $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty}$ can be arranged into three pairwise disjoint subsequences $\{\lambda_k^{(j)}\}_{-\infty, k \neq 0}^{\infty}$, $j = 1, 2, 3$ enumerated in the following way: $\lambda_{-k}^{(1)} = -\lambda_k^{(1)}$, $\lambda_{-k}^{(2)} = -\lambda_k^{(2)}$, $\lambda_{-k}^{(3)} = -\lambda_k^{(3)}$ and $\lambda_k^{(j)} \leq \lambda_{k+1}^{(j)}$ for $j = 1, 2, 3$, such that $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty} = \bigcup_{j=1}^3 \{\lambda_k^{(j)}\}_{-\infty, k \neq 0}^{\infty}$, and

$$\lambda_k^{(1)} = \frac{k\pi}{a} + \varepsilon_k^{(1)}, \quad (23)$$

$$\lambda_k^{(j)} = \frac{k\pi + (-1)^j \sin^{-1} \sqrt{\frac{2}{3}}}{a} + \varepsilon_k^{(j)}, \quad j = 2, 3, \quad (24)$$

where $\varepsilon_k^{(j)} = o(1)$, as $k \rightarrow \infty$ for $j = 1, 2, 3$. It is not difficult to see that

$$\varepsilon_k^{(j)} = O\left(\frac{1}{k}\right), \quad k \rightarrow \infty, \quad j = 1, 2, 3. \quad (25)$$

Substituting (23) into $\lambda_k^{(1)}\Phi(\lambda_k^{(1)}) = 0$, then from (15) and taking into account that the function $\omega_1(\lambda)$ is bounded on the real axis by the Paley-Wiener theorem, we obtain

$$\begin{aligned} \lambda_k^{(1)}\Phi(\lambda_k^{(1)}) &= (-1)^k \left(2 \sin \varepsilon_k^{(1)} a - 3 \sin^3 \varepsilon_k^{(1)} a \right) \\ &\quad + (-1)^k a (2B_1 + 2B_2 + 3B_3 + \beta) \frac{\sin^2 \varepsilon_k^{(1)} a \cos \varepsilon_k^{(1)} a}{k\pi} \\ &\quad - (-1)^k a (B_1 + B_2) \frac{\cos^3 \varepsilon_k^{(1)} a}{k\pi} + \frac{a\omega_1(\lambda_k^{(1)})}{k\pi} + O\left(\frac{1}{k^2}\right) \end{aligned}$$

$$= (-1)^k 2 \sin \varepsilon_k^{(1)} a + O\left(\frac{1}{k}\right) = 0, \quad k \rightarrow \infty.$$

This yields $\sin \varepsilon_k^{(1)} a = O\left(\frac{1}{k}\right)$. Thus, $\varepsilon_k^{(1)} = O\left(\frac{1}{k}\right)$. Similarly, we can show that $\varepsilon_k^{(j)} = O\left(\frac{1}{k}\right)$ for $j = 2, 3$. Substituting (23) into the equation $\lambda_k^{(1)} \Phi(\lambda_k^{(1)}) = 0$ where $\Phi(\lambda)$ is given by (15), by expanding the left-hand side of resulting equation in power series and taking into account (25) and $\{\omega_1(\lambda_k^{(1)})\}_{-\infty, k \neq 0}^\infty \in l_2$ (see [16, Lemma 1.4.3]), we obtain

$$2\varepsilon_k^{(1)} a - \frac{a(B_1 + B_2)}{k\pi} + \frac{\tau_k}{k} = 0,$$

where $\{\tau_k\}_{-\infty, k \neq 0}^\infty \in l_2$. Solving this equation we get (21). In the same way, we get (22).

To compare necessary conditions on a sequence to be the spectrum of the boundary-value problem L_0 with the sufficient condition which will be obtained in Section 4, we need more precise asymptotics.

Theorem 3.4. *Let $q_j(x) \in W_2^1(0, a)$ for $j=1,2,3$. Then the subsequences of Theorem 3.3 behave asymptotically as follows:*

$$\lambda_k^{(1)} = \frac{k\pi}{a} + \frac{B_1 + B_2}{2k\pi} + \frac{\gamma_k^{(1)}}{k^2}, \quad (26)$$

$$\lambda_k^{(j)} = \frac{k\pi + (-1)^j \sin^{-1} \sqrt{\frac{2}{3}}}{a} + \frac{3B_1 + 3B_2 + 6B_3 + 2\beta}{12k\pi} + \frac{\gamma_k^{(j)}}{k^2}, \quad j = 2, 3, \quad (27)$$

where $\{\gamma_k^{(j)}\}_{-\infty, k \neq 0}^\infty \in l_2$ for $j = 1, 2, 3$.

Proof. If $q_j(x) \in W_2^1(0, a)$, twice integrating by parts the formulas of [7, p. 18] and [16, p. 9], we obtain

$$u_j(a, \lambda) = \frac{\sin \lambda a}{\lambda} - B_j \frac{\cos \lambda a}{\lambda^2} + D_j \frac{\sin \lambda a}{\lambda^3} + \frac{\varrho_j(\lambda)}{\lambda^3}, \quad j = 1, 2, \quad (28)$$

$$u_3(a, \lambda) = \cos \lambda a + B_3 \frac{\sin \lambda a}{\lambda} + D_3 \frac{\cos \lambda a}{\lambda^2} + \frac{\varrho_3(\lambda)}{\lambda^2}, \quad (29)$$

$$u'_j(a, \lambda) = \cos \lambda a + B_j \frac{\sin \lambda a}{\lambda} + D'_j \frac{\cos \lambda a}{\lambda^2} + \frac{\sigma_j(\lambda)}{\lambda^2}, \quad j = 1, 2, \quad (30)$$

$$u'_3(a, \lambda) = -\lambda \sin \lambda a + B_3 \cos \lambda a + D'_3 \frac{\sin \lambda a}{\lambda} + \frac{\sigma_3(\lambda)}{\lambda}, \quad (31)$$

where $D_j, D'_j, j = 1, 2, 3$ are constants and $\varrho_j(\lambda), \sigma_j(\lambda), j = 1, 2, 3$ are entire functions of class L^a . Substituting (28)-(31) into (11) we obtain

$$\begin{aligned} \Phi(\lambda) = & \frac{2 \sin \lambda a - 3 \sin^3 \lambda a}{\lambda} + (2B_1 + 2B_2 + 3B_3 + \beta) \frac{\sin^2 \lambda a \cos \lambda a}{\lambda^2} \\ & - (B_1 + B_2) \frac{\cos^3 \lambda a}{\lambda^2} + E_1 \frac{\sin^3 \lambda a}{\lambda^3} + E_2 \frac{\cos^2 \lambda a \sin \lambda a}{\lambda^3} + \frac{\omega_3(\lambda)}{\lambda^3}, \end{aligned} \tag{32}$$

where E_1, E_2 are constants and $\omega_3(\lambda) \in L^{3a}$. Substituting (21) into the equation $\lambda_k^{(1)} \Phi(\lambda_k^{(1)}) = 0$ where $\Phi(\lambda)$ is given by (32) and by expanding the left-hand side of resulting equation in power series, we get (26). Analogously, we obtain (27). Theorem 3.4 is proved.

Remark 3.5. Under the conditions of Theorem 3.4, the spectra $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$ of the boundary-value problems L_j for $j = 1, 2, 3$ behave asymptotically as follows (see [16, p. 75]):

$$\nu_k^{(j)} = \frac{k\pi}{a} + \frac{B_j}{\pi k} + \frac{\delta_k^{(j)}}{k^2}, \quad j = 1, 2, \tag{33}$$

$$\nu_k^{(3)} = \frac{\pi(k - \frac{1}{2})}{a} + \frac{B_3}{\pi(k - \frac{1}{2})} + \frac{\delta_k^{(3)}}{k^2}, \tag{34}$$

where $\{\delta_k^{(j)}\}_{-\infty, k \neq 0}^\infty \in l_2$ for $j = 1, 2, 3$.

For investigation of direct and inverse spectral problems, methods of the theory of entire and meromorphic functions are widely used. For this reason, we give several notation and definitions for what follows.

If $\Omega \subseteq \mathbb{C}$ is an open set, we denote by $\mathcal{H}(\Omega)$ the set of all functions which are analytic in Ω and by $\mathcal{M}(\Omega)$ the set of all functions meromorphic in Ω .

Definition 3.6. ([25]) Let $\mathcal{K} \subseteq \mathcal{M}(\mathbb{C})$ and let $\varphi, \psi \in \mathcal{H}(\mathbb{C})$.

- i. The pair (φ, ψ) is called a 1- \mathcal{K} -pair, if $\psi^{-1}\varphi \in \mathcal{K}$ and φ and ψ have no common zeros.
- ii. Let $n \in \mathbb{N}$ and $n \geq 2$. The pair (φ, ψ) is called an n - \mathcal{K} -pair, if $\psi^{-1}\varphi \in \mathcal{K}$, there exist 1- \mathcal{K} -pairs $(\varphi_1, \psi_1), \dots, (\varphi_n, \psi_n)$ such that

$$\psi = \prod_{i=1}^n \psi_i, \quad \varphi = \sum_{i=1}^n \left(\varphi_i \prod_{\substack{j=1 \\ j \neq i}}^n \psi_j \right),$$

and no representation of this kind is possible with less than n many 1- \mathcal{K} -pairs.

Definition 3.7.([25]) A function $f \in \mathcal{H}(\mathbb{C} \setminus \mathbb{R})$ is said to be of Nevanlinna class \mathcal{N} if

- i. $f(z) = \overline{f(\bar{z})}$ for $z \in \mathbb{C} \setminus \mathbb{R}$;
- ii. $\text{Im } f(z) \geq 0$ for $\text{Im } z > 0$.

Definition 3.8.([25]) The class \mathcal{N}^{ep} of essentially positive Nevanlinna functions is defined as the set of all functions $f \in \mathcal{N}$ which are analytic in $\mathbb{C} \setminus [0, \infty)$ with possible exception of finitely many poles. Moreover, the class \mathcal{N}_-^{ep} is defined as the set of all functions $f \in \mathcal{N}$ such that for some $b \in \mathbb{R}$ we have $f \in \mathcal{H}(\mathbb{C} \setminus [b, \infty))$ and $f(z) \leq 0$ for $z \in (-\infty, b)$.

It is easy to check that $\mathcal{N}_-^{ep} \subseteq \mathcal{N}^{ep}$.

Definition 3.9. ([15]) An entire function $\omega(z)$ of exponential type $\sigma > 0$ is said to be a function of sine-type if it satisfies the following conditions:

- i. all the zeros of $\omega(z)$ lie in a strip $|\text{Im } z| < h < \infty$;
- ii. for some h_1 and all $z \in \{\lambda : \text{Im } z = h_1\}$, the following equalities hold:

$$0 < m \leq |\omega(z)| \leq M < \infty;$$

- iii. the type of $\omega(z)$ in the lower half-plane coincides with that in the upper half-plane.

Let us introduce the entire functions

$$\varphi_j(z) = -u'_j(a, \sqrt{z}) - \frac{\beta}{3} u_j(a, \sqrt{z}), \quad j = 1, 2, 3, \quad (35)$$

$$\psi_j(z) = u_j(a, \sqrt{z}), \quad j = 1, 2, 3, \quad (36)$$

$$\varphi(z) = -\Phi(\sqrt{z}), \quad \psi(z) = \Psi(\sqrt{z}). \quad (37)$$

Using (11) and (12) we obtain

$$\varphi(z) = \sum_{i=1}^3 \left(\varphi_i(z) \prod_{\substack{j=1 \\ j \neq i}}^3 \psi_j(z) \right), \quad \psi(z) = \prod_{j=1}^3 \psi_j(z) \quad (38)$$

and consequently

$$\frac{\varphi(z)}{\psi(z)} = \sum_{j=1}^3 \frac{\varphi_j(z)}{\psi_j(z)}. \quad (39)$$

Lemma 3.10.

1. The zeros of the functions $\varphi_j(z)$ and $\psi_j(z)$ ($j = 1, 2, 3$) are real;
 2. The functions $\varphi_j(z)$ and $\psi_j(z)$ ($j = 1, 2, 3$) have no common zeros.
- Proof.* The zeros of $\varphi_j(z)$, $j = 1, 2, 3$ coincide with the squares of the eigenvalues of the boundary-value problems

$$\begin{cases} -y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), & x \in [0, a], \\ y_j(0) = y_j'(a) + \frac{\beta}{3}y_j(a) = 0, & j = 1, 2, \end{cases}$$

$$\begin{cases} -y_3''(x) + q_3(x)y_3(x) = \lambda^2 y_3(x), & x \in [0, a], \\ y_3'(0) - h y_3(0) = y_3'(a) + \frac{\beta}{3}y_3(a) = 0, \end{cases}$$

respectively, and the zeros of $\psi_j(z)$ coincide with the squares of the eigenvalues of the boundary-value problems L_j , $j = 1, 2, 3$, respectively. These problems are self-adjoint and it follows from [20, Part I, Theorem 3] that the squares of their eigenvalues are real. Assertion 1 is proved. To prove assertion 2, let z_0 be a common zero of $\varphi_j(z)$ and $\psi_j(z)$. Using to Lagrange identity(see [20, Part II, p. 50]) for solutions $u_j(a, \sqrt{z})$ and $u_j(a, \sqrt{z_0})$ of (1) we obtain

$$(z - z_0) \int_0^a u_j(x, \sqrt{z})u_j(x, \sqrt{z_0})dx = (u_j(x, \sqrt{z})u_j'(x, \sqrt{z_0}) - u_j'(x, \sqrt{z})u_j(x, \sqrt{z_0})) \Big|_0^a = \varphi_j(z)\psi_j(z_0) - \varphi_j(z_0)\psi_j(z).$$

For $z \rightarrow z_0$ we get

$$\int_0^a u_j^2(x, \sqrt{z_0})dx = \dot{\varphi}_j(z_0)\psi_j(z_0) - \varphi_j(z_0)\dot{\psi}_j(z_0) = 0,$$

where $\dot{\varphi}_j(z) = \frac{d}{dz}\varphi_j(z)$ and $\dot{\psi}_j(z) = \frac{d}{dz}\psi_j(z)$. This implies that $u_j(x, \sqrt{z_0}) \equiv 0$ which is a contradiction. Therefore, $\varphi_j(z)$ and $\psi_j(z)$ have no common zeros.

Lemma 3.11. The functions $\frac{\varphi_j(z)}{\psi_j(z)}$, $j = 1, 2, 3$ and $\frac{\varphi(z)}{\psi(z)}$ are of the Nevanlinna class \mathcal{N} .

Proof. Let $j \in \{1, 2, 3\}$. Using the Lagrange identity for the solution $u_j(a, \sqrt{z})$ of (1), we have

$$\left(\overline{u_j'(x, \sqrt{z})}u_j(x, \sqrt{z}) - u_j'(x, \sqrt{z})\overline{u_j(x, \sqrt{z})} \right) \Big|_0^a = 2i\text{Im } z \int_0^a |u_j(x, \sqrt{z})|^2 dx. \quad (40)$$

Since

$$\operatorname{Im} \left(\overline{u'_j(a, \sqrt{z})} u_j(a, \sqrt{z}) - u'_j(a, \sqrt{z}) \overline{u_j(a, \sqrt{z})} \right) = -2|u_j(a, \sqrt{z})|^2 \operatorname{Im} \frac{u'_j(a, \sqrt{z})}{u_j(a, \sqrt{z})},$$

then (40) yields

$$-\operatorname{Im} \frac{u'_j(a, \sqrt{z})}{u_j(a, \sqrt{z})} = \operatorname{Im} z \frac{\int_0^a |u_j(x, \sqrt{z})|^2 dx}{|u_j(a, \sqrt{z})|^2}, \quad \operatorname{Im} z \neq 0.$$

Thus,

$$\operatorname{Im} \left(-\frac{u'_j(a, \sqrt{z})}{u_j(a, \sqrt{z})} \right) \geq 0 \quad \text{for } \operatorname{Im} z > 0$$

and consequently

$$\operatorname{Im} \frac{\varphi_j(z)}{\psi_j(z)} = \operatorname{Im} \left(-\frac{u'_j(a, \sqrt{z})}{u_j(a, \sqrt{z})} - \frac{\beta}{3} \right) \geq 0 \quad \text{for } \operatorname{Im} z > 0. \quad (41)$$

Also, according to Lemma 3.10 the zeros of $\varphi_j(z)$ and $\psi_j(z)$ are real and hence $\frac{\varphi_j(z)}{\psi_j(z)} \in \mathcal{H}(\mathbb{C} \setminus \mathbb{R})$. Therefore $\frac{\varphi_j(z)}{\psi_j(z)} \in \mathcal{N}$. Now it follows from (39) and (41) that $\frac{\varphi(z)}{\psi(z)} \in \mathcal{H}(\mathbb{C} \setminus \mathbb{R})$ and

$$\operatorname{Im} \frac{\varphi(z)}{\psi(z)} = \sum_{j=1}^3 \operatorname{Im} \frac{\varphi_j(z)}{\psi_j(z)} \geq 0 \quad \text{for } \operatorname{Im} z > 0.$$

Consequently $\frac{\varphi(z)}{\psi(z)} \in \mathcal{N}$. Lemma 3.11 is proved.

Lemma 3.12. *The functions $\frac{\varphi_j(z)}{\psi_j(z)}$, $j = 1, 2, 3$ and $\frac{\varphi(z)}{\psi(z)}$ are of the class \mathcal{N}_-^{ep} .*

Proof. By virtue of the formulas (17)-(20) we get

$$u_j(a, \sqrt{z}) = \frac{e^{\sqrt{|z|}a}}{2\sqrt{|z|}} (1 + o(1)), \quad z \rightarrow -\infty, \quad j = 1, 2,$$

$$u_3(a, \sqrt{z}) = \frac{e^{\sqrt{|z|}a}}{2} (1 + o(1)), \quad z \rightarrow -\infty,$$

$$u'_j(a, \sqrt{z}) = \frac{e^{\sqrt{|z|}a}}{2} (1 + o(1)), \quad z \rightarrow -\infty, \quad j = 1, 2,$$

$$u'_3(a, \sqrt{z}) = \frac{\sqrt{|z|}e^{\sqrt{|z|}a}}{2} (1 + o(1)), \quad z \rightarrow -\infty.$$

Using these asymptotics we obtain from (35) and (36)

$$\frac{\varphi_j(z)}{\psi_j(z)} = -\sqrt{|z|}(1 + o(1)), \quad z \rightarrow -\infty, \quad j = 1, 2, 3,$$

and consequently

$$\lim_{z \rightarrow -\infty} \frac{\varphi_j(z)}{\psi_j(z)} = -\infty, \quad j = 1, 2, 3. \quad (42)$$

It follows from Lemma 3.10, Lemma 3.11 and (42) that there exist real numbers $b_j \in \mathbb{R}$, $j = 1, 2, 3$ such that

$$\frac{\varphi_j(z)}{\psi_j(z)} \in \mathcal{N} \cap H(\mathbb{C} \setminus [b_j, \infty)), \quad \frac{\varphi_j(z)}{\psi_j(z)} < 0 \quad \text{for } z \in (-\infty, b_j). \quad (43)$$

Therefore, $\frac{\varphi_j(z)}{\psi_j(z)} \in \mathcal{N}_-^{ep}$ for $j = 1, 2, 3$.

Now using (39) and (43) and Lemma 3.11 we conclude that

$$\frac{\varphi(z)}{\psi(z)} \in \mathcal{N} \cap \mathcal{H}(\mathbb{C} \setminus [b, \infty)), \quad \frac{\varphi(z)}{\psi(z)} = \sum_{j=1}^3 \frac{\varphi_j(z)}{\psi_j(z)} < 0 \quad \text{for } z \in (-\infty, b),$$

where $b = \min\{b_1, b_2, b_3\}$. Thus, $\frac{\varphi(z)}{\psi(z)} \in \mathcal{N}_-^{ep}$. Lemma 3.12 is proved.

Theorem 3.13 *The sequences $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$ and $\{\kappa_k\}_{-\infty, k \neq 0}^\infty$ satisfy the following conditions:*

1. $0 < \lambda_1 < \kappa_1 \leq \lambda_2 \leq \kappa_2 \leq \dots \leq \lambda_k \leq \kappa_k \leq \dots$ ($\lambda_{-k} = -\lambda_k$, $\kappa_{-k} = -\kappa_k$);
2. $\kappa_k = \lambda_{k+1}$ if and only if $\lambda_{k+1} = \kappa_{k+1}$ for $k \in \mathbb{N}$;
3. The maximal multiplicity of κ_k is 3.

Proof. Denote $\mathring{\mathcal{N}}_-^{ep} := \mathcal{M}(\mathbb{C}) \cap \mathcal{N}_-^{ep}$. The functions $u_j(a, \sqrt{z})$ and $u'_j(a, \sqrt{z})$ are entire in z and hence in view of Lemma 3.12, $\frac{\varphi_j(z)}{\psi_j(z)} \in \mathring{\mathcal{N}}_-^{ep}$ for $j = 1, 2, 3$ and $\frac{\varphi(z)}{\psi(z)} \in \mathring{\mathcal{N}}_-^{ep}$. Also, by Lemma 3.10 the functions $\varphi_j(z), \psi_j(z)$ have no common zeros. Therefore the pairs (φ_j, ψ_j) , $j = 1, 2, 3$ are $1\text{-}\mathring{\mathcal{N}}_-^{ep}$ -pairs and consequently, in view of (38) the pair (φ, ψ) is an $m\text{-}\mathring{\mathcal{N}}_-^{ep}$ -pair with some $m \leq 3$ (see Definition 3.6). On the other hand by virtue of (37), the squares of the zeros of $\Phi(\lambda)$ and $\Psi(\lambda)$ coincide with the zeros of $\varphi(z)$ and $\psi(z)$, respectively. Now the assertions of Theorem 3.13 immediately follows from [25, Corollary 4.6].

4. INVERSE PROBLEM

In the present section, we study the problem of reconstruction of the potential $q(x) = [q_j(x)]_{j=1,2,3}$ and the real numbers h, β from the given spectral characteristics. Let us denote by Q the class of sets $\{[q_j(x)]_{j=1,2,3}, h, \beta\}$ which satisfy the following conditions:

- i. $q_j(x)$, $j = 1, 2, 3$ are real-valued functions from $L_2(0, a)$;
- ii. $h, \beta \in \mathbb{R}$;
- iii. the operator A constructed via (4), (5) is strictly positive.

Theorem 4.1. *Let the following conditions be satisfied :*

1. Three sequences $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, $j = 1, 2, 3$ of real numbers are such that

i. $\nu_{-k}^{(j)} = -\nu_k^{(j)}$, $\nu_k^{(j)} < \nu_{k+1}^{(j)}$, $\nu_k^{(j)} \neq 0$ for all $k \in \mathbb{N}$ and $j = 1, 2, 3$;

ii. $\{\nu_k^{(i)}\}_{-\infty, k \neq 0}^\infty \cap \{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty = \emptyset$ for $i \neq j$, $i, j = 1, 2, 3$;

iii.

$$\nu_k^{(j)} = \frac{\pi k}{a} + \frac{B_j}{\pi k} + \frac{\delta_k^{(j)}}{k^2}, \quad j = 1, 2, \quad (44)$$

$$\nu_k^{(3)} = \frac{\pi(k - \frac{1}{2})}{a} + \frac{B_3}{\pi(k - \frac{1}{2})} + \frac{\delta_k^{(3)}}{k^2}, \quad (45)$$

where B_j are real constants, $B_i \neq B_j$ for $i \neq j$ and $\{\delta_k^{(j)}\}_{-\infty, k \neq 0}^\infty \in l_2$ for $j = 1, 2, 3$.

2. A sequence $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$ of real numbers ($\lambda_{-k} = -\lambda_k$, $\lambda_k \leq \lambda_{k+1}$, $\lambda_k \neq 0$ for all $k \in \mathbb{N}$) can be represented as the union of three pairwise disjoint subsequences $\{\lambda_k\}_{-\infty, k \neq 0}^\infty = \bigcup_{j=1}^3 \{\lambda_k^{(j)}\}_{-\infty, k \neq 0}^\infty$ ($\lambda_{-k}^{(1)} = -\lambda_k^{(1)}$, $\lambda_{-k}^{(2)} = -\lambda_k^{(2)}$, $\lambda_{-k}^{(3)} = -\lambda_k^{(3)}$ and $\lambda_k^{(j)} \leq \lambda_{k+1}^{(j)}$ for $j = 1, 2, 3$) which behave asymptotically as follows:

$$\lambda_k^{(1)} = \frac{k\pi}{a} + \frac{B_1 + B_2}{2k\pi} + \frac{\gamma_k^{(1)}}{k^2}, \quad (46)$$

$$\lambda_k^{(j)} = \frac{k\pi + (-1)^j \sin^{-1} \sqrt{\frac{2}{3}}}{a} + \frac{B_0}{k\pi} + \frac{\gamma_k^{(j)}}{k^2}, \quad j = 2, 3, \quad (47)$$

where B_0 is a real constant and $\{\gamma_k^{(j)}\}_{-\infty, k \neq 0}^\infty \in l_2$ for $j = 1, 2, 3$.

3. The sequences $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$ and $\{\kappa_k\}_{-\infty}^\infty := \bigcup_{j=1}^3 \{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty \cup \{0\}$ ($\kappa_{-k} = -\kappa_k$, $\kappa_k < \kappa_{k+1}$) interlace in the following strict sense:

$$\cdots < \kappa_{-2} < \lambda_{-2} < \kappa_{-1} < \lambda_{-1} < \kappa_0 = 0 < \lambda_1 < \kappa_1 < \lambda_2 < \kappa_2 < \cdots. \quad (48)$$

Then there exists a unique set $\{[q_j(x)]_{j=1,2,3}, h, \beta\} \in Q$ such that the sequence $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$ coincides with the spectrum of the boundary-value problem L_0 , where

$\beta = 6B_0 - \frac{3}{2}B_1 - \frac{3}{2}B_2 - 3B_3$, $h = B_3 - (1/2) \int_0^a q_3(x) dx$ and the sequences $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, $j = 1, 2, 3$ coincides with the spectra of the boundary-value problems L_j , $j = 1, 2, 3$, respectively.

Proof. Denote by

$$\begin{aligned} \{\rho_k^{(0)}\}_{-\infty, k \neq 0}^\infty &:= \left\{ \frac{\pi k - \xi}{a} \right\}_{-\infty, k \neq 0}^\infty \cup \left\{ \frac{\pi k + \xi}{a} \right\}_{-\infty, k \neq 0}^\infty, \quad \xi := \sin^{-1} \sqrt{\frac{2}{3}}, \\ \{\rho_k\}_{-\infty, k \neq 0}^\infty &:= \{\lambda_k^{(2)}\}_{-\infty, k \neq 0}^\infty \cup \{\lambda_k^{(3)}\}_{-\infty, k \neq 0}^\infty. \end{aligned}$$

It is possible to enumerate $\{\rho_k^{(0)}\}_{-\infty, k \neq 0}^\infty$ and $\{\rho_k\}_{-\infty, k \neq 0}^\infty$ in the proper way ($\rho_{-k}^{(0)} = -\rho_k^{(0)}$, $\rho_k^{(0)} < \rho_{k+1}^{(0)}$ and $\rho_{-k} = -\rho_k$, $\rho_k \leq \rho_{k+1}$). Let us construct the following entire functions:

$$u_j(\lambda) = a \prod_1^\infty \left(\frac{a^2}{\pi^2 k^2} (\nu_k^{(j)2} - \lambda^2) \right), \quad j = 1, 2, \quad (49)$$

$$u_3(\lambda) = \prod_1^\infty \left(\frac{a^2}{\pi^2 (k - 1/2)^2} (\nu_k^{(3)2} - \lambda^2) \right), \quad (50)$$

$$\phi_1(\lambda) = a \prod_1^\infty \left(\frac{a^2}{\pi^2 k^2} (\lambda_k^{(1)2} - \lambda^2) \right), \quad (51)$$

$$\phi_2(\lambda) = 2 \prod_1^\infty \left(\frac{1}{\rho_k^{(0)2}} (\rho_k^2 - \lambda^2) \right). \quad (52)$$

Using [21, Lemma 2.1], we obtain

$$u_j(\lambda) = \frac{\sin \lambda a}{\lambda} - B_j \frac{\cos \lambda a}{\lambda^2} + F_j \frac{\sin \lambda a}{\lambda^3} + \frac{f_j(\lambda)}{\lambda^3}, \quad j = 1, 2, \quad (53)$$

where F_j , $j = 1, 2$ are constants and $f_j(\lambda) \in L^a$ for $j = 1, 2$. In the same way as [21, Lemma 2.1] we can prove that

$$u_3(\lambda) = \cos \lambda a + B_3 \frac{\sin \lambda a}{\lambda} + F_3 \frac{\cos \lambda a}{\lambda^2} + \frac{f_3(\lambda)}{\lambda^2}, \quad (54)$$

$$\phi_1(\lambda) = \frac{\sin \lambda a}{\lambda} - \left(\frac{B_1 + B_2}{2} \right) \frac{\cos \lambda a}{\lambda^2} + G_1 \frac{\sin \lambda a}{\lambda^3} + \frac{g_1(\lambda)}{\lambda^3}, \quad (55)$$

$$\phi_2(\lambda) = 2 - 3 \sin^2 \lambda a + 3B_0 \frac{\sin 2\lambda a}{\lambda} + G_2 \frac{2 - 3 \sin^2 \lambda a}{\lambda^2} + \frac{g_2(\lambda)}{\lambda^2}, \quad (56)$$

where F_3 , G_j , $j = 1, 2$ are constants and $f_3(\lambda), g_1(\lambda) \in L^a$ and $g_2(\lambda) \in L^{2a}$.

Let us set

$$X_k^{(j)} := \nu_k^{(j)} \left(\frac{\phi_1(\nu_k^{(j)})\phi_2(\nu_k^{(j)})}{u_i(\nu_k^{(j)})u_3(\nu_k^{(j)})} - \cos \nu_k^{(j)} a - B_j \frac{\sin \nu_k^{(j)}}{\nu_k^{(j)}} \right), \quad i, j = 1, 2, i \neq j, \quad (57)$$

$$X_k^{(3)} := \left(\frac{\phi_1(\nu_k^{(3)})\phi_2(\nu_k^{(3)})}{u_1(\nu_k^{(3)})u_2(\nu_k^{(3)})} + \nu_k^{(3)} \sin \nu_k^{(3)} a - B_3 \cos \nu_k^{(3)} a \right). \quad (58)$$

where the numbers B_j , $j = 1, 2, 3$ can be determined by

$$B_j = \lim_{k \rightarrow \infty} k\pi \left(\nu_k^{(j)} - \frac{\pi k}{a} \right), \quad j = 1, 2, \quad (59)$$

$$B_3 = \lim_{k \rightarrow \infty} \pi \left(k - \frac{1}{2} \right) \left(\nu_k^{(3)} - \frac{\pi \left(k - \frac{1}{2} \right)}{a} \right). \quad (60)$$

It is clear that $X_{-k}^{(j)} = -X_k^{(j)}$ for $j = 1, 2$ and $X_{-k}^{(3)} = X_k^{(3)}$. To complete the proof we need the following lemma.

Lemma 4.2.

$$\{X_k^{(j)}\}_{-\infty, k \neq 0}^{\infty} \in l_2 \quad \text{for } j = 1, 2, 3. \quad (61)$$

Proof. Substituting (44) into (53)-(56), we obtain

$$u_2(\nu_k^{(1)}) = (-1)^k \frac{a^2(B_1 - B_2)}{\pi^2 k^2} + \frac{\zeta_k^{(1)}}{k^3}, \quad (62)$$

$$u_3(\nu_k^{(1)}) = (-1)^k \left(1 - \frac{a^2 B_1^2}{2\pi^2 k^2} + \frac{a^2 B_1 B_3}{\pi^2 k^2} + \frac{a^2 F_3}{\pi^2 k^2} \right) + \frac{\zeta_k^{(2)}}{k^2}, \quad (63)$$

$$\phi_1(\nu_k^{(1)}) = (-1)^k \frac{a^2(B_1 - B_2)}{\pi^2 k^2} + \frac{\zeta_k^{(3)}}{k^3}, \quad (64)$$

$$\phi_2(\nu_k^{(1)}) = 2 + \frac{\zeta_k^{(4)}}{k}, \quad (65)$$

where $\{\zeta_k^{(j)}\}_{-\infty, k \neq 0}^{\infty} \in l_2$ for $j = \overline{1, 4}$. Also, Using (44), we obtain the asymptotic relation

$$\cos \nu_k^{(1)} a + B_1 \frac{\sin \nu_k^{(1)}}{\nu_k^{(1)}} = (-1)^k \left(1 + \frac{\eta_k}{k} \right), \quad (66)$$

where $\{\eta_k\}_{-\infty, k \neq 0}^\infty \in l_2$. If we substitute (62)-(66) into (57), then we conclude that $\{X_k^{(1)}\}_{-\infty, k \neq 0}^\infty \in l_2$. Analogously we can show that $\{X_k^{(2)}\}_{-\infty, k \neq 0}^\infty \in l_2$. We show that $\{X_k^{(3)}\}_{-\infty, k \neq 0}^\infty \in l_2$. Let us substitute (45) into (53)-(56). We obtain

$$u_j(\nu_k^{(3)}) = \frac{(-1)^{k+1}}{\nu_k^{(3)}} \left(1 + \frac{\zeta_k^{(j)}}{k} \right), \quad j = 1, 2, \quad (67)$$

$$\phi_1(\nu_k^{(3)}) = \frac{(-1)^{k+1}}{\nu_k^{(3)}} \left(1 + \frac{\zeta_k^{(3)}}{k} \right), \quad (68)$$

$$\phi_2(\nu_k^{(3)}) = -1 + \frac{\zeta_k^{(4)}}{k}, \quad (69)$$

where $\{\zeta_k^{(j)}\}_{-\infty, k \neq 0}^\infty \in l_2$ for $j = \overline{1, 4}$. Furthermore, taking into account (45), we have

$$\nu_k^{(3)} \sin \nu_k^{(3)} a - B_3 \cos \nu_k^{(3)} a = (-1)^{k+1} \nu_k^{(3)} \left(1 + \frac{\eta'_k}{k} \right), \quad (70)$$

where $\{\eta'_k\}_{-\infty, k \neq 0}^\infty \in l_2$. Using (67)-(70) in (58), the assertion of Lemma 4.2 for $j = 3$ follows.

Now Since the functions $\lambda u_j(\lambda)$, $j = 1, 2$ and $u_3(\lambda)$ are sine-type functions (see Definition 3.9) and by virtue of (44), (45) and (48), $\inf_{k \neq p} |\nu_k^{(j)} - \nu_p^{(j)}| > 0$ for $j = 1, 2, 3$ (and hence the zeros of $\lambda u_j(\lambda)$, $j = 1, 2$ and $u_3(\lambda)$ are simple), the Lagrange interpolation series

$$\lambda u_j(\lambda) \sum_{\substack{-\infty \\ k \neq 0}}^{\infty} \frac{X_k^{(j)}}{\left. \frac{d\lambda u_j(\lambda)}{d\lambda} \right|_{\lambda=\nu_k^{(j)}} (\lambda - \nu_k^{(j)})}, \quad j = 1, 2 \quad (71)$$

and

$$u_3(\lambda) \sum_{\substack{-\infty \\ k \neq 0}}^{\infty} \frac{X_k^{(3)}}{\left. \frac{du_3(\lambda)}{d\lambda} \right|_{\lambda=\nu_k^{(3)}} (\lambda - \nu_k^{(3)})} \quad (72)$$

constructed on the basis of the sequences $\{X_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, define functions $\varepsilon_j(\lambda) \in L^a$, $j = 1, 2, 3$, respectively (see [14, Theorem A]). Using these functions, we define the even entire functions

$$v_j(\lambda) = \cos \lambda a + B_j \frac{\sin \lambda a}{\lambda} + \frac{\varepsilon_j(\lambda)}{\lambda}, \quad j = 1, 2, \quad (73)$$

$$v_3(\lambda) = -\lambda \sin \lambda a + B_3 \cos \lambda a + \varepsilon_3(\lambda). \quad (74)$$

It follows directly from (71) and (72) that $\varepsilon_j(\nu_k^{(j)}) = X_k^{(j)}$ for $j = 1, 2, 3$ and hence

$$v_j(\nu_k^{(j)}) = \frac{\phi_1(\nu_k^{(j)})\phi_2(\nu_k^{(j)})}{u_i(\nu_k^{(j)})u_3(\nu_k^{(j)})}, \quad i, j = 1, 2, i \neq j, \quad (75)$$

$$v_3(\nu_k^{(3)}) = \frac{\phi_1(\nu_k^{(3)})\phi_2(\nu_k^{(3)})}{u_1(\nu_k^{(3)})u_2(\nu_k^{(3)})}. \quad (76)$$

Let us denote by $\{\mu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$ the set of zeros of the functions $v_j(\lambda)$, $j = 1, 2, 3$, respectively. These sets are symmetric with respect to the real axis and to the imaginary axis. Hence, we number the zeros in the proper way: $\mu_{-k}^{(j)} = -\mu_k^{(j)}$, $\operatorname{Re} \mu_k^{(j)} \leq \operatorname{Re} \mu_{k+1}^{(j)}$ for all $k \in \mathbb{N}$ and the multiplicities are taken into account (we shall prove that all $\mu_k^{(j)2}$ are real and all $\mu_k^{(j)}$ are simple except for $\mu_1^{(j)}$, if $\mu_1^{(j)} = \mu_{-1}^{(j)} = 0$). It follows from (73) and (74) that

$$\mu_k^{(j)} = \frac{\pi(k - \frac{1}{2})}{a} + \frac{B_j}{\pi(k - \frac{1}{2})} + \frac{\theta_k^{(j)}}{k}, \quad j = 1, 2, \quad (77)$$

$$\mu_k^{(3)} = \frac{(k-1)\pi}{a} + \frac{B_3}{k\pi} + \frac{\theta_k^{(3)}}{k}, \quad (78)$$

where $\{\theta_k^{(j)}\}_{-\infty, k \neq 0}^\infty \in l_2$ for $j = 1, 2, 3$.

Proposition 4.3. *The following inequalities are valid:*

$$\mu_1^{(j)2} < \nu_1^{(j)2} < \mu_2^{(j)2} < \nu_2^{(j)2} < \dots, \quad j = 1, 2, 3. \quad (79)$$

Proof. In the same way as proof of [22, Proposition 2.3], we can show that

$$\begin{aligned} (-1)^k \frac{\phi_1(\nu_k^{(j)})\phi_2(\nu_k^{(j)})}{u_i(\nu_k^{(j)})u_3(\nu_k^{(j)})} &> 0, \quad i, j = 1, 2, i \neq j, \\ (-1)^k \frac{\phi_1(\nu_k^{(3)})\phi_2(\nu_k^{(3)})}{u_1(\nu_k^{(3)})u_2(\nu_k^{(3)})} &> 0. \end{aligned}$$

From these inequalities and (75) and (76), it follows that

$$(-1)^k v_j(\nu_k^{(j)}) > 0, \quad j = 1, 2, 3. \quad (80)$$

Let $j \in \{1, 2, 3\}$ be fixed. It follows from (80) that between consecutive $\nu_k^{(j)}$'s there is an odd number (with account of multiplicities) of $\mu_k^{(j)}$'s. Suppose that there are

three or more of them between $\nu_k^{(j)}$ and $\nu_{k+1}^{(j)}$. Then comparing (77) and (78) with (44) and (45), we conclude that there are no $\mu_p^{(j)}$'s between some $\nu_{k'}^{(j)}$ and $\nu_{k'+1}^{(j)}$ where $k \neq k'$, a contradiction. Thus, $\nu_1^{(j)2} < \mu_2^{(j)2} < \nu_2^{(j)2} < \dots$. If $v_j(0) > 0$, then $0 < \mu_1^{(j)} < \nu_1^{(j)}$. If $v_j(0) = 0$, then $\mu_1^{(j)} = 0$. If $v_j(0) < 0$, then $\mu_1^{(j)}$ is a pure imaginary number and hence $\mu_1^{(j)2} < \nu_1^{(j)2}$. Proposition 4.3 is proved.

Let $j \in \{1, 2\}$. It follows from (79) and the asymptotic relations (44) and (77) that the sequences $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$ and $\{\mu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$ satisfy the conditions of [16, Theorem 3.4.1]. Thus, there exists a unique real-valued function $q_j(x) \in L_2(0, a)$ such that $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$ and $\{\mu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$ are the spectra of the boundary-value problems L_j and L'_j , respectively. An algorithm for the reconstruction of this potential $q_j(x)$ is as follows(see [16, Section 3.4]): Without loss of generality, let us assume that $\mu_1^{(j)2} > 0$, otherwise we apply a shift. The function

$$e_j(\lambda) = e^{i\lambda a} (v_j(\lambda) - i\lambda u_j(\lambda))$$

is the so-called Jost function of the corresponding prolonged Sturm–Liouville problem on the semi-axis:

$$-y_j''(x) + \tilde{q}_j(x)y_j(x) = \lambda^2 y_j(x), \quad x \in [0, \infty), \quad y_j(0) = 0,$$

where

$$\tilde{q}_j(x) = \begin{cases} q_j(x) & \text{if } x \in [0, a] \\ 0 & \text{if } x \in (a, \infty). \end{cases}$$

Then we construct the S-function of the problem on the semi-axis:

$$S_j(\lambda) = \frac{e_j(-\lambda)}{e_j(\lambda)}$$

and the function

$$F_j(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - S_j(\lambda)) e^{i\lambda x} d\lambda.$$

Solving the Marchenko integral equation

$$K_j(x, t) + F_j(x + t) + \int_x^\infty K_j(x, s) F_j(x + s) ds = 0, \quad t > x$$

we find the unique solution $K_j(x, t)$ and

$$q_j(x) = -2 \frac{dK_j(x, x)}{dx}, \quad x \in [0, a].$$

The two sequences $\{\nu_k^{(3)}\}_{-\infty, k \neq 0}^\infty$ and $\{\mu_k^{(3)}\}_{-\infty, k \neq 0}^\infty$ satisfy (due to (45), (78) and (79)) the conditions of [7, Theorem 1.5.4]. Thus, there exists a unique real-valued function $q_3(x) \in L_2(0, a)$ and a unique real number h such that $\{\nu_k^{(3)}\}_{-\infty, k \neq 0}^\infty$ and $\{\mu_k^{(3)}\}_{-\infty, k \neq 0}^\infty$ are the spectra of the boundary-value problems L_3 and L'_3 , respectively. Below we give the algorithm of recovering of $q_3(x)$ as it is described in [7, Section 1.5]. Calculate the so-called weight numbers $\{\alpha_k\}_1^\infty$ of the problem L'_3 by

$$\alpha_k = \frac{1}{2\mu_k^{(3)}} \dot{v}_3(\mu_k^{(3)}) u_3(\mu_k^{(3)}),$$

where $\dot{v}_3(\lambda) = \frac{d}{d\lambda} v_3(\lambda)$. If $\mu_1^{(3)} = 0$, then $\dot{v}_3(\mu_1^{(3)}) = 0$ and we set $\alpha_1 = \frac{1}{2} \ddot{v}_3(0) u_3(0)$ where $\ddot{v}(\lambda) = \frac{d^2}{d\lambda^2} v_3(\lambda)$. Construct the function

$$F(x, t) = \sum_{k=1}^{\infty} \left(\frac{\cos \mu_k^{(3)} x \cos \mu_k^{(3)} t}{\alpha_k} - \frac{\cos(k-1)x \cos(k-1)t}{\alpha_k^0} \right),$$

where

$$\alpha_k^0 = \begin{cases} \frac{a}{2}, & k > 1 \\ a, & k = 1. \end{cases}$$

Then using the unique solution of the Gel'fand-Levitan integral equation

$$K_3(x, t) + F(x, t) + \int_0^x K_3(x, s) F(s, t) ds = 0, \quad 0 \leq t \leq x \leq a,$$

we find

$$q_3(x) = 2 \frac{dK_3(x, x)}{dx}, \quad h = K_3(0, 0) = B_3 - \frac{1}{2} \int_0^a q_3(x) dx.$$

To find β , we compare (27) and (47) and set

$$\beta = 6B_0 - \frac{3}{2}B_1 - \frac{3}{2}B_2 - 3B_3, \quad (81)$$

where B_0 can be determined by

$$B_0 = \lim_{k \rightarrow \infty} k\pi \left(\lambda_k^{(2)} - \frac{k\pi + \sin^{-1} \sqrt{\frac{2}{3}}}{a} \right). \quad (82)$$

Now we prove that the spectrum of the problem L_0 which is generated by the obtained $[q_j(x)]_{j=1,2,3}$, h and β coincides with $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$. Due to [16, Theorem 3.4.1] and [7, Theorem 1.5.4], the spectra of the problems L_j , $j = 1, 2, 3$ which are generated by the obtained $[q_j(x)]_{j=1,2,3}$ and h coincide with $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, $j = 1, 2, 3$,

respectively. The function $u_j(a, \lambda)$ ($j = 1, 2, 3$) where $u_j(x, \lambda)$ is the solution of (1) with obtained $[q_j(x)]_{j=1,2,3}$ and h which satisfy the initial conditions (3), coincide with $u_j(\lambda)$, since they have the same zeros and the same asymptotics. Also, since $u'_j(a, \lambda)$ and $v_j(\lambda)$ ($j = 1, 2, 3$) have the same asymptotics and according to [16, Theorem 3.4.1] and [7, Theorem 1.5.4] have the same zeros, hence they coincide. Thus, the values of the function $\Phi(\lambda)$ (defined by (11)) at $\lambda = \nu_k^{(j)}$ coincide with

$$\phi_1(\nu_k^{(j)})\phi_2(\nu_k^{(j)})$$

for all $k \in \mathbb{Z} \setminus \{0\}$ and all $j = 1, 2, 3$, i.e., with the corresponding values of the function $\phi_1(\lambda)\phi_2(\lambda)$. This implies that the entire function

$$\Delta(\lambda) := \Phi(\lambda) - \phi_1(\lambda)\phi_2(\lambda) \quad (83)$$

of exponential type $3a$ can be represented as follows:

$$\Delta(\lambda) = t(\lambda) \prod_{j=1}^3 u_j(a, \lambda), \quad (84)$$

where $t(\lambda)$ is an entire function. Using (16), (54) and (55) we have

$$\Delta(\lambda) = t(\lambda) \left(\frac{\sin^2 \lambda a \cos \lambda a}{\lambda^2} - (B_1 + B_2) \frac{\cos^2 \lambda a \sin \lambda a}{\lambda^3} + B_3 \frac{\sin^3 \lambda a}{\lambda^3} + \frac{\omega_1(\lambda)}{\lambda^3} \right), \quad (85)$$

$$\begin{aligned} \phi_1(\lambda)\phi_2(\lambda) = & \frac{2 \sin \lambda a - 3 \sin^3 \lambda a}{\lambda} + (6B_0 + \frac{1}{2}B_1 + \frac{1}{2}B_2) \frac{\sin^2 \lambda a \cos \lambda a}{\lambda^2} \\ & - (B_1 + B_2) \frac{\cos^3 \lambda a}{\lambda^2} + E'_1 \frac{\sin^3 \lambda a}{\lambda^3} + E'_2 \frac{\cos^2 \lambda a \sin \lambda a}{\lambda^3} + \frac{\omega_2(\lambda)}{\lambda^3}, \end{aligned} \quad (86)$$

where E'_1, E'_2 are constants and $\omega_1(\lambda), \omega_2(\lambda) \in L^{3a}$. Substituting (32), (85) and (86) into (83) and using (81), we obtain

$$\begin{aligned} t(\lambda)(\lambda \sin^2 \lambda a \cos \lambda a - (B_1 + B_2) \cos^2 \lambda a \sin \lambda a + B_3 \sin^3 \lambda a + \omega_1(\lambda)) \\ = (E_1 - E'_1) \sin^3 \lambda a + (E_2 - E'_2) \cos^2 \lambda a \sin \lambda a + \omega_3(\lambda), \end{aligned} \quad (87)$$

where $\omega_3(\lambda) \in L^{3a}$. Since the functions $\sin^3 \lambda a, \cos^2 \lambda a \sin \lambda a, \omega_1(\lambda)$ and $\omega_3(\lambda)$ are bounded on the real axis, hence relation (87) implies that $t(\lambda) \equiv 0$ and from (83), it follows that $\Phi(\lambda) = \phi_1(\lambda)\phi_2(\lambda)$. Consequently, the sequence $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty}$ coincides with the spectrum of the boundary-value problem L_0 . The operator A constructed

by (4), (5) using the obtained $[q_j(x)]_{j=1,2,3}$, h and β is strictly positive, because it is self-adjoint and its spectrum is positive. The uniqueness of the solution of the inverse problem follows from the fact that formulas (71) and (72) establishes one-to-one correspondence between l_2 and L^a (see [14, Theorem A]). The proof of Theorem 3.1 is finished.

Remark 4.4. If condition 1(ii) of Theorem 4.1 fails, i.e., the sets $\{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty$, $j = 1, 2, 3$ are not pairwise disjoint (consequently, the condition 3 fails too), either the uniqueness or the existence result of mentioned theorem can also fails, for the same reasons as in the case of three spectra (see [9, 21]). If the sequences $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$ and $\{\kappa_k\}_{-\infty}^\infty = \bigcup_{j=1}^3 \{\nu_k^{(j)}\}_{-\infty, k \neq 0}^\infty \cup \{0\}$ are not pairwise disjoint and satisfy the statements of Theorem 4.1, then the solution of the inverse problem exists but is not unique.

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REFERENCES

- [1] V. Adamyan. *Scattering matrices for microschemes*, Oper. Theory Adv. Appl., 59:1–10, 1992.
- [2] G. Berkolaiko, E. Bogomolny and J. Keating, *Star graphs and Seba biliards*, J. Phys. A: Math. Gen., 34 (2001), 335–350.
- [3] R. P. BOAS, *Entire Functions*. Academic Press, New York, 1954.
- [4] R. Carlson, *Hills equation for a homogeneous tree*, Electron. J. Differential Equations, 23 (1997), 1–30.
- [5] S. Currie, *Spectral Theory of Differential Operators on Graphs*. PhD Thesis, University of the Witwatersrand, Johannesburg, 2006.
- [6] M. D. Faddeev and B. S. Pavlov, *Model of free electrons and the scattering problem*, Theoret. and Math. Phys., 55(1983), 485–492.
- [7] G. Freiling and V. A. Yurko, *Inverse Sturm–Liouville Problems and Their Applications*, Nova Science Publishers, New York, 2001.
- [8] N. Gerasimenko and B. Pavlov, *Scattering problems on non-compact graphs*, *Theoretical and Mathematical Physics*, 74 (1988), 230–240.
- [9] F. Gesztesy and B. Simon, *On the determination of a potential from three spectra*, In *Differential operators and spectral theory*, volume 189 of Amer. Math. Soc. Transl. Ser. 2, pages 85–92, Amer. Math. Soc., Providence, RI, 1999.

- [10] V. Kostykin and R. Schrader, *Kirchhoffs rule for quantum wires*, J. Phys. A: Math. Gen., 32(1999), 595–630.
- [11] P. Kuchment, *Differential and psuedo-differential operators on graphs as models of mesoscopic systems*, In Analysis and Applications—ISAAC 2001 (Berlin), Volume 10 of Int. Soc. Anal. Appl. Comput., pages 7–30. Kluwer Acad. Publ., Dordrecht, 2003.
- [12] P. Kuchment, *Graph models for waves in thin structures*, Waves Random Media, 12 (2002), R1-R24.
- [13] B. Ja. Levin, *Lectures on Entire Functions*. Translations of Mathematical Monographs Vol. 150, Amer. Math. Soc., Providence, RI, 1996.
- [14] B. Ja. Levin and Yu. I. Lyubarskii, *Interpolation by entire functions of special classes and related expansions in series of exponents*, Izv. Acad. Sci. USSR, Ser. Mat., 39(3)(1975), 657–702(in Russian); English transl.: Math. USSR Izv., 9(3)(1975), 621–6629.
- [15] B.Ya. Levin and I.V. Ostrovskii, *On small perturbations of the set of roots of a sinus-type function*, Izv. Akad. Nauk USSR Ser. Mat., 43:87–110, 1979(in Russian); English transl.: *Math. USSR Izv.*, 14(1) (1980), 79–101.
- [16] V.A. Marchenko, *Sturm–Liouville Operators and Applications*. Naukova Dumka, Kiev, 1977; English transl., Birkhauser, Basel, 1986.
- [17] Yu. B. Melnikov and B. S. Pavlov, *Two-body scattering on a graph and application to simple Nanoelectronic devices*, J. Math. Phys., 36 (1995), 2813–2838.
- [18] Yu. B. Melnikov and B. S. Pavlov, *Scattering on graphs and one-dimensional approximations to N -dimensional Schrodinger operators*, J. Math. Phys., 42(3) (2001), 1202–1228.
- [19] E. Montrol, *Quantum theory on a network*, J. Math. Phys., 11 (1970), 635–648.
- [20] N. Naimark, *Linear Differential Operators, Parts I and II*. Frederick Ungar Publishing Co., New York, 1968.
- [21] V. Pivovarchik, *An inverse Sturm–Liouville problem by three spectra*, Integral Equations Operator Theory, 34 (1999), 234–243.
- [22] V. Pivovarchik, *Inverse problem for the Sturm–Liouville equation on a simple graph*, SIAM J. Math. Anal., 32(4) (2000), 801–819.
- [23] V. Pivovarchik, *Direct and inverse three-point Sturm–Liouville problem with parameter dependent boundary conditions*, Asymptotic Anal., 26 (2001), 219–238.
- [24] V. Pivovarchik, *Inverse problem for the Sturm–Liouville equation on a star-shaped graph*, Math. Nachr., 280(13-14) (2007), 1595-1619.
- [25] V. Pivovarchik and H. Woracek, *Sums of Nevanlinna functions and differential equations on star-shaped graphs*, Oper. Matrices, 3(4) (2009), 451-501.
- [26] Yu. V. Pokornyi and A. V. Borovskikh, *Differential equations on networks*

(*geometric graphs*), J. Math. Sci. (NY), 119 (2004), 691–718.

[27] Yu. Pokornyi and V. Pryadiev, *The qualitative Sturm–Liouville theory on spatial networks*, J. Math. Sci. (NY), 119 (2004), 788–835.

[28] K. Ruedenberg and C.W. Scherr, *Free-electron network model for conjugated systems: I. Theory*, J. Chem. Phys., 21 (1953), 1565–1581.

[29] C. F. Yang, *Inverse spectral problems for the Sturm–Liouville operator on a d -star graph*, J. math. Anal. Appl., 365(2)(2010), 742–749.

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