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GROUPS, COMPLETE ℓ-GROUPOIDS, AND COHEN-MACAULAY RINGS

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Abstract. In this paper, R is a commutative ring with an identity, $L(R)$ is the lattice of ideals of R, and $L(G)$ is the lattice of all subgroups of a finite group G. It is shown that if $L(R)$ is a principal lattice, P is cl-groupoid with zero, and G is a finite cyclic group, then $R[P]$ and $R[L(G)]$ are Cohen-Macaulay rings.

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1. Introduction

The basic concept of abstract commutative ideal theory is the concept of Noether lattice which was introduced by R. P. Dilworth [3] as an abstraction of the lattice of ideals of a Noetherian ring. Recall in [3], a multiplicative lattice is a complete lattice L with a commutative, associative multiplication which distributes over arbitrary joins and such that the largest element I of L is the identity for the multiplication. Basically, an element E of a multiplicative lattice L is said to be meet (join-) principal if $(A \wedge (B : E))E = (AE) \wedge B$ (if $(BE \vee A) : E = B \vee (A : E)$) for all A and B in L. A principal element is an element that is both meet-principal and join-principal. L is called principal lattice when each of its elements is principal. Cohen-Macaulay property is one of the most important notion in the commutative algebra. Local ring R is said Cohen-Macaulay when so is R as an R-module. A Noetherian ring (which may not be local) R is said to be Cohen-Macaulay when its localization at any maximal ideal is Cohen-Macaulay local. In Section 2, relationship Brouwerian lattice, po-groupoid, and Cohen-Macaulay ring is considered. In end section it is shown that, if $L(R)$ is principal lattice and G is a finite cyclic group, then $R[L(G)]$ is Cohen-Macaulay.

2.Residuation And Brouwerian lattice And Cohen-Macaulay

A number of papers are devoted to the subject of guaranteeing the distributivity of lattice by imposing equations on a fixed generating set. A lattice L is distributive when it satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$. In a bounded distributive lattice, a is a complement of b iff $a \wedge b = 0$ and $a \vee b = 1$. Let $a \in [b,\ c]$; x is a relative complement of a in $[b,\ c]$ iff $a \wedge x = b$ and $a \vee x = c$. A complement lattice is a bounded lattice in which every element has a complement. A relatively complemented lattice is a lattice in which every element has a relative complement in any interval containing it. A Boolean lattice is a complemented distributive lattice. A Boolean algebra is Boolean lattice in which 0, 1, and $'$ are considered to be operation. In any Boolean algebra A, a' is the largest element x such that $x \wedge a = 0$. More generally, $a \wedge x \leq b$ if and only if $a \wedge x \wedge b' = 0$, that is $(a \wedge b') \wedge x = 0$ or $x \leq (a \wedge b')' = b \vee a'$. Hence, given $a, b \in A$, there exists a largest element $c = b \vee a'$ such that $a \wedge c \leq b$. Brouwer and Heyting characterized an important generalization of Boolean algebra through an extension of the preceding property.
More generally, let L be a lattice with 0; an element a^* is a pseudocomplement of a $(\in L)$ iff $a \wedge a^* = 0$, and $a \wedge x = 0$ implies that $x \leq a^*$.

Definition 2.1. A Brouwerian lattice is a lattice L in which, for any given elements a and b, the set of all $x \in L$ such that $a \wedge x \leq b$ contains a greatest element $b : a$, the relative pseudo-complement of a in b.

Theorem 2.1. *Any Brouwerian lattice is distributive.*

Proof. Given a, b, c form $d = (a \wedge b) \vee (a \wedge c)$, and consider $d : a$. Since $a \wedge b \leq d$ and $a \wedge c \leq d$, we have $b \leq d : a$ and $c \leq d : a$. Hence $b \wedge c \leq d : a$, and so $a \wedge (b \vee c) \leq a \wedge (d : a) \leq d = (a \wedge b) \vee (a \wedge c)$.

Definition 2.2. A *po-groupoid*(or *m-poset*) is a poset M with a binary multiplication which satisfies the isotonicity condition $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for all $a, b, x \in M$.

Example 2.1. In any ring R, the additive subgroups $X, Y, Z, \ldots$ form an m-poset with zero under inclusion, if XY is defined as the set of all finite sums $\sum x_i y_i$ $(x_i \in X, y_i \in Y)$.

Definition 2.3. If M is a lattice with a multiplication and $a(b \vee c) = ab \vee ac$, $(a \vee b)c = ac \vee bc$, for all $a, b, c \in M$ holds, then M is called an m-lattice or *ℓ-groupoid.*

One of the most important concepts in the theory of ℓ-groupoids is that of residual, defined as follows.

Example 2.2. In any ring R, the two-sided ideals form a residuated lattice, under the multiplication of Example 2.1.

Corollary 2.1. *A lattice L is a residuated lattice, when xy is defined as $x \wedge y$, if and only if it is a Brouwerian lattice.*

Most residuated lattices arising in applications are complete, and satisfy the infinite distributive laws $a(\vee b_\beta) = \vee(ab_\beta)$ and $(a_\alpha)b = \vee(a_\alpha b)$. A lattice L is called complete if $\sup H$ and $\inf H$ exists for all $H \subseteq P$. This leads us to make the following definition.

Definition 2.4. A complete ℓ-groupoid, or *cl-groupoid*, is a complete lattice with a binary multiplication satisfying $a(\vee b_\beta) = \vee(ab_\beta)$ and $(a_\alpha)b = \vee(a_\alpha b)$. A cl-groupoid with associative multiplication is called a *cl-semigroup*, if it has a 1, it is called a *cl-monoid*.

The modules of a ring (Example 2.1, 2.2) constitute a typical cl-groupoid; $a(\vee b_\beta) = \vee(ab_\beta)$ and $(a_\alpha)b = \vee(a_\alpha b)$ follows from the fact that the operations involved are finitely(binary); we omit the verification. An ℓ-groupoid is not just a *po-groupoid* which is lattice under its partial ordering relation: products must also be distributive on joins.

Theorem 2.2. *Let R be a commutative ring with an identity such that $L(R)$ be a principal lattice. If P is cl-groupoid with zero, then $R[P]$ is Cohen-Macaulay.*

Proof. If $L(R)$ is a principal lattice, then R is a Noetherian multiplication ring. So the ring R is Cohen-Macaulay ring (see [5], [4] and [6]). On the other hand, if R is Cohen-Macaulay ring and P is a distributive lattice, then $R[P]$ is Cohen-Macaulay (see [2]). In particular, any cl-groupoid with zero is residuated. Thus, it is a Brouwerian lattice. Completing the proof of Theorem 2.1 and Corollary 2.1.

3.Subgroup lattices And Cohen-Macaulay

Note that, a ring is called a multiplication ring, if every ideal of R is product of two ideals. Let M be a finitely generated module over a Noetherian ring R. We say that $x \in R$ is an M-regular element, if $xg = 0$ for $g \in M$ implies $g = 0$, in the other words, if x is not a zero-divisor on M. A sequence $x_1, \cdots x_r$ of elements of the ring R, is called an M-regular sequence or simply an M-sequence if the following conditions are satisfied:

1. x_i is an $M/(x_1, \cdots, x_{i-1})M$-regular element for $i = 1, \ldots, r$;
2. $M/(x_1, \ldots, x_r)M \neq 0$.

Suppose $I \subseteq R$ is an ideal with $IM \neq M$. The *depth* of I on M is maximal length of an M-regular sequence in I, denoted by $depth(I, M)$. If R is a local ring with a unique maximal ideal $\mathbf{m}$, we write, $depth(\mathbf{m})$, for $depth(\mathbf{m}, M)$. Let R be a Noetherian local ring. A finitely generated R-module M, is a Cohen-Macaulay module, if $depth(M) = dim(M)$. If R itself is a Cohen-Macaulay module, then it is called a Cohen-Macaulay ring.

Let $\sum$ consist of the subgroups of any group G, and let $\leq$ mean set-inclusion. Then $\sum$ is a complete lattice, with $H \wedge K = H \cap K$ (set-intersection), and $H \vee K$ the least subgroup in $\sum$ containing H and K (which is not their set-theoretical union). We now turn our attention to the lattice $L(G)$ of all subgroups of G.

Theorem 3.1. The lattice $L(G)$ of all subgroups of a finite group is distributive if and only if G is cyclic.

Proof. ($\Longrightarrow$) Let Z_r be a finite cyclic group of order r, with generator a. Then [1] every subgroup of Z_r is cyclic, with generator a^s for some $s|r$. Hence the lattice of positive integers, under the relation $m|n$. This shows that the lattice of all subgroups of any finite cyclic group is distributive.

($\Longleftarrow$) In group G, let A, B, C be cyclic subgroups generated by a, b, and $c = ab$, respectively. If $(A \vee B) \wedge C = (A \wedge C) \wedge (B \wedge C$,then $ab = ba$. Hence, if $L(G)$ is distributive, then G must be the direct product of cyclic groups of prime-power order $q_1 = p^{k_1}, \ldots, q_r = p_r^{k_r}$; with generators $a_1, \ldots, a_r$. If two were equal, then G would contain two elements $b_i = a_i^{q_i/p}$ and $b_j = a_j^{q_i/p}, p = p_i = p_j$. These would generate an elementary Abelian group whose subgroup-lattice was not distributive. Hence, the p_i are all distinct; but in this case, $a = a_1 a_2 \ldots a_r$ is of order $q_1 q_2 \ldots q_r$, and generates G, which is therefore cyclic.

Corollary 3.1. *If R is Cohen-Macaulay ring and G is a finite cyclic group, then $R[L(G)]$ is Cohen-Macaulay.*

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