Acta Universitatis Apulensis No. 31/2012
ISSN: 1582-5329 pp. 327-330

A SIMPLE METRIC FOR FINITE DIMENSIONAL VECTOR SPACES

Oskar Maria Baksalary and Götz Trenkler

Abstract. A new metric for subspaces of a finite dimensional vector space V is identified. The metric is determined by the dimensions of $\mathcal{M} + \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are subspaces of V. Some properties of the metric are derived as well.

2000 *Mathematics Subject Classification*: 15A03

1. Introduction

It is well known that the *Euclidean distance* $\|\mathbf{x} - \mathbf{y}\|$ between $\mathbf{x}$ and $\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are elements of an n-dimensional complex vector space $\mathbb{C}_{n,1}$, defines a metric in the set $\mathbb{C}_{n,1}$. Recall that the symbol $\|\mathbf{z}\|$ stands for the Euclidean norm of $\mathbf{z} \in \mathbb{C}_{n,1}$ given by $\|\mathbf{z}\| = \sqrt{\mathbf{z}^*\mathbf{z}}$, where $\mathbf{z}^*$ is the conjugate transpose of $\mathbf{z}$. Suppose now that instead of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}_{n,1}$, we have two vector spaces $\mathcal{M}, \mathcal{N} \subseteq \mathbb{C}_{n,1}$. A relevant question occurs, how can a distance between $\mathcal{M}$ and $\mathcal{N}$ be meaningfully defined?

Several concepts which serve as a measure of the distance between subspaces were introduced in the literature. One of the most often used metrics in the set of all subspaces of $\mathbb{C}_{n,1}$ is called simply *distance* (or *gap*), and for $\mathcal{M}, \mathcal{N} \subseteq \mathbb{C}_{n,1}$ is defined by

$$\mathrm{dist}(\mathcal{M}, \mathcal{N}) = \|\mathbf{P}_{\mathcal{M}} - \mathbf{P}_{\mathcal{N}}\|_2, \tag{1}$$

where $\|\cdot\|_2$ denotes the spectral norm of a matrix argument, and $\mathbf{P}_{\mathcal{M}}$ and $\mathbf{P}_{\mathcal{N}}$ are the orthogonal projectors onto $\mathcal{M}$ and $\mathcal{N}$, respectively; see [3, p. 387]. For a complex $m \times n$ matrix $\mathbf{N}$, the spectral norm satisfies $\|\mathbf{N}\|_2 = \sqrt{\lambda_{\max}(\mathbf{N}^*\mathbf{N})}$, with $\lambda_{\max}(\mathbf{N}^*\mathbf{N})$ being the largest eigenvalue of $\mathbf{N}^*\mathbf{N}$.

Another known subspace metric is obtained from (1) by replacing the spectral norm with the Frobenius norm, which leads to the so called *Frobenius distance*

$$\mathrm{dist}_F(\mathcal{M}, \mathcal{N}) = \|\mathbf{P}_{\mathcal{M}} - \mathbf{P}_{\mathcal{N}}\|_F. \tag{2}$$

The Frobenius norm is for $m \times n$ complex matrix $\mathbf{N}$ defined by $\|\mathbf{N}\|_F = \sqrt{\operatorname{tr}(\mathbf{N}^*\mathbf{N})}$, where $\operatorname{tr}(\cdot)$ is trace of a matrix argument.

As was pointed out in [1, Section 1], the metrics defined in (1) and (2) may coincide in some cases and differ in other. Consider, for instance, the subspaces $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ of $\mathbb{R}_{2,1}$ specified as

$$\mathcal{L} = \operatorname{span}\{\begin{pmatrix}1\\1\end{pmatrix}\}, \quad \mathcal{M} = \mathbb{R}_{2,1}, \quad \text{and} \quad \mathcal{N} = \operatorname{span}\{\begin{pmatrix}1\\0\end{pmatrix}\}. \tag{3}$$

Then $\operatorname{dist}(\mathcal{L},\mathcal{M}) = \operatorname{dist}_F(\mathcal{L},\mathcal{M}) = 1$, but $\operatorname{dist}(\mathcal{L},\mathcal{N}) = \frac{1}{\sqrt{2}}$ and $\operatorname{dist}_F(\mathcal{L},\mathcal{N}) = 1$, i.e., $\operatorname{dist}(\mathcal{L},\mathcal{N}) \neq \operatorname{dist}_F(\mathcal{L},\mathcal{N})$; for details see [1, Section 1].

2. Main result

Yet another subspace metric is introduced in the theorem below, being the main result of the present note. The definition of this metric does not refer to any matrix norm and is based exclusively on the notion of a dimension of a subspace. It takes integer values only and seems to be simpler than the distance and Frobenius distance. However, probably the most important advantage of the metric is that its applicability is not restricted to the space $\mathbb{C}_{n,1}$, for it is valid in any vector space of a finite dimension. With this respect, the theorem below generalizes [1, Theorem 7], where the validity of the metric was restricted to particular vector spaces.

Theorem. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a finite dimensional vector space V. Moreover, let* $\dim(.)$ *denote dimension of a subspace argument. Then*

$$\mathrm{d}(\mathcal{M},\mathcal{N}) := \dim(\mathcal{M}+\mathcal{N}) - \dim(\mathcal{M}\cap\mathcal{N})$$

is a metric for the subspaces of V.

Proof. It has to be shown that $\mathrm{d}(\mathcal{M},\mathcal{N})$ enjoys the following properties: (i) $\mathrm{d}(\mathcal{M},\mathcal{N}) > 0$ if $\mathcal{M} \neq \mathcal{N}$, $\mathrm{d}(\mathcal{M},\mathcal{M}) = 0$; (ii) $\mathrm{d}(\mathcal{M},\mathcal{N}) = \mathrm{d}(\mathcal{N},\mathcal{M})$; and (iii) $\mathrm{d}(\mathcal{M},\mathcal{N}) \leqslant \mathrm{d}(\mathcal{M},\mathcal{L}) + \mathrm{d}(\mathcal{L},\mathcal{N})$, where $\mathcal{L} \subseteq V$; see e.g., [3, §13.1].

Since $\mathcal{M} \neq \mathcal{N}$ implies $\dim(\mathcal{M}+\mathcal{N}) > \dim(\mathcal{M}\cap\mathcal{N})$, it is clear that $\mathrm{d}(\mathcal{M},\mathcal{N})$ is necessarily nonnegative. Moreover, when $\mathcal{M} = \mathcal{N}$, then $\dim(\mathcal{M}+\mathcal{N}) = \dim(\mathcal{M}\cap\mathcal{N})$, ensuring that $\mathrm{d}(\mathcal{M},\mathcal{M}) = 0$. Further, the property (ii) is visibly fulfilled by the fact that subspace sum and intersection are both commutative. To show that $\mathrm{d}(\mathcal{M},\mathcal{N})$ satisfies also the third property of a metric, we rewrite the inequality in (iii) as

$$\dim(\mathcal{M}+\mathcal{N})-\dim(\mathcal{M}\cap\mathcal{N}) \leqslant \dim(\mathcal{M}+\mathcal{L})-\dim(\mathcal{M}\cap\mathcal{L})+\dim(\mathcal{L}+\mathcal{N})-\dim(\mathcal{L}\cap\mathcal{N}).$$

Hence, on account of the *subspace dimension theorem*, which for $\mathcal{J}, \mathcal{K} \subseteq V$ reads

$$\dim(\mathcal{J} + \mathcal{K}) = \dim(\mathcal{J}) + \dim(\mathcal{K}) - \dim(\mathcal{J} \cap \mathcal{K}), \tag{4}$$

(see e.g., [4, Theorem 1.6]), it is seen that the inequality in (iii) can be equivalently expressed as

$$\dim(\mathcal{M} \cap \mathcal{L}) + \dim(\mathcal{L} \cap \mathcal{N}) \leqslant \dim(\mathcal{L}) + \dim(\mathcal{M} \cap \mathcal{N}). \tag{5}$$

Since $\dim[(\mathcal{M} \cap \mathcal{L}) + (\mathcal{L} \cap \mathcal{N})] \leqslant \dim(\mathcal{L})$ and $\dim[(\mathcal{M} \cap \mathcal{L}) \cap (\mathcal{L} \cap \mathcal{N})] \leqslant \dim(\mathcal{M} \cap \mathcal{N})$, the validity of (5) follows from (4) by taking $\mathcal{J} = \mathcal{M} \cap \mathcal{L}$ and $\mathcal{K} = \mathcal{L} \cap \mathcal{N}$. □

In view of (4), it is clear that a convenient alternative expression for $\mathrm{d}(\mathcal{M}, \mathcal{N})$ is $\mathrm{d}(\mathcal{M}, \mathcal{N}) = \dim(\mathcal{M}) + \dim(\mathcal{N}) - 2\dim(\mathcal{M} \cap \mathcal{N})$. Note also that for the subspaces specified in (3), we have $\mathrm{d}(\mathcal{L}, \mathcal{M}) = 1$ and $\mathrm{d}(\mathcal{L}, \mathcal{N}) = 2$, that is $\mathrm{d}(\mathcal{L}, \mathcal{M})$ coincides with the distance (and the Frobenius distance), whereas $\mathrm{d}(\mathcal{L}, \mathcal{N})$ is different from both $\mathrm{dist}(\mathcal{L}, \mathcal{N})$ and $\mathrm{dist}_F(\mathcal{L}, \mathcal{N})$.

3. Supplementary results

The note is concluded with two remarks each of which identifies an interesting property of the metric introduced in Theorem.

Remark 1. Let V be a finite dimensional inner product vector space. It is known that $\mathcal{M}, \mathcal{N} \subseteq V$ satisfy $C(\mathcal{M}, \mathcal{N}) = C[\mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}, \mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}]$, where $C(\mathcal{M}, \mathcal{N})$ denotes the angle between $\mathcal{M}$ and $\mathcal{N}$ and the superscript "$\perp$" stands for the orthogonal complement of a subspace; see [2, Lemma 9.5]. It turns out that an analogous property characterizes also $\mathrm{d}(\mathcal{M}, \mathcal{N})$, i.e., that

$$\mathrm{d}(\mathcal{M}, \mathcal{N}) = \mathrm{d}[\mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}, \mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}].$$

This characteristic follows from an observation that when $\mathcal{J}, \mathcal{K} \subseteq V$ are such that $\mathcal{K} \subseteq \mathcal{J}$, then $\dim(\mathcal{J} \cap \mathcal{K}^{\perp}) = \dim(\mathcal{J}) - \dim(\mathcal{K})$.

Remark 2. Let $\mathcal{L}$ be a subspace of a finite dimensional vector space V and let $\mathcal{L}^{\circ}$ be the *annihilator* of $\mathcal{L}$, that is the set consisting of all those linear functionals on V which map every element of $\mathcal{L}$ into zero; see [4, §3.2]. It can be proved that $\mathcal{M}, \mathcal{N} \subseteq V$ satisfy

$$\mathrm{d}(\mathcal{M}, \mathcal{N}) = \mathrm{d}(\mathcal{M}^{\circ}, \mathcal{N}^{\circ}).$$

To derive this identity, one may refer to known facts that $\mathcal{M}, \mathcal{N} \subseteq V$ satisfy $\dim(\mathcal{M}) + \dim(\mathcal{M}^{\circ}) = \dim(V)$, $(\mathcal{M} + \mathcal{N})^{\circ} = \mathcal{M}^{\circ} \cap \mathcal{N}^{\circ}$, and $(\mathcal{M} \cap \mathcal{N})^{\circ} = \mathcal{M}^{\circ} + \mathcal{N}^{\circ}$; see [4, Theorems 3.1 and 3.2].

Acknowledgements

The final version of this paper was prepared while the first author was visiting the Faculty of Statistics at the Dortmund University of Technology. The financial supports from the Alexander von Humboldt Foundation as well as from the project POKL run at the Faculty of Physics of the Adam Mickiewicz University are gratefully acknowledged. The author is also very thankful to the Faculty of Statistics for provision of excellent facilities during the stay.

References

[1] O.M. Baksalary and G. Trenkler, *On a subspace metric based on a matrix rank*, Linear Algebra and its Applications 432, (2010), 1475-1491.

[2] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer, New York, 2001.

[3] I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, SIAM, Philadelphia, 2006.

[4] G.C. Shephard, *Vector Spaces of Finite Dimension*, Oliver and Boyd, Edinburgh, 1966.

Oskar Maria Baksalary
Faculty of Physics
Adam Mickiewicz University
ul. Umultowska 85, PL 61-614 Poznań, Poland
email: *obaksalary@gmail.com*

Götz Trenkler
Faculty of Statistics
Dortmund University of Technology
Vogelpothsweg 87, D-44221 Dortmund, Germany
email: *trenkler@statistik.tu-dortmund.de*