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SUBORDINATION RESULTS FOR A CLASS OF MULTIVALENT NON-BAZILEVIC ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract. In this paper, by making use of the principle of subordination, we introduce a class of multivalent non-Bazlevic analytic functions defined by linear operator. Various results as subordination, superordination properties, distortion theorems and inequality properties are proved.

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1.Introduction

Let H be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be subclass of H consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... \ (z \in U).$$

Also, let $\mathcal{A}(p)$ denote the subclass of H consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{1.1}$$

We write $\mathcal{A}(1) = \mathcal{A}_1$. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, or $g(z)$ is superordinate to $f(z)$, written symbolically, $f \prec g$ in U or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$ such that $f(z) = g(\omega(z))$ $(z \in U)$. Futher more, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [9]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp^{'}(z); z\right) \prec h(z), \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp^{'}(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp^{'}(z); z\right), \tag{1.3}$$

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant. For further properties of subordination and superordination see [4] and [9].

For functions f given by (1.1) and $g \in A(p)$ given by $g(z) = z^p + \sum\limits_{k=p+1}^{\infty} b_k z^k$ $(p \in \mathbb{N})$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

For functions $f, g \in A(p)$, we define the linear operator $D^m_{\lambda,p} : A(p) \to A(p)(\lambda \geqslant 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ by:

$$D^0_{\lambda,p}(f * g)(z) = (f * g)(z),$$

$$\begin{aligned} D^1_{\lambda,p}(f * g)(z) &= D_{\lambda,p}(f * g)(z) = (1-\lambda)(f * g)(z) + \frac{\lambda z}{p}((f * g)(z))' \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{p+\lambda(k-p)}{p} a_k b_k z^k \end{aligned}$$

and (in general)

$$D^m_{\lambda,p}(f * g)(z) = D_{\lambda,p}(D^{m-1}_{\lambda,p}(f * g)(z))$$

$$= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\lambda(k-p)}{p} \right)^m a_k b_k z^k \quad , \lambda \geqslant 0 \,. \tag{1.4}$$

From (1.4), we can easily deduce that

$$\frac{\lambda z}{p} \left(D^m_{\lambda,p}(f*g)(z) \right)' = D^{m+1}_{\lambda,p}(f*g)(z) - (1-\lambda) D^m_{\lambda,p}(f*g)(z) \; (\lambda > 0). \tag{1.5}$$

The operator $D^m_{\lambda,p}(f*g)$ was introduced and studied by Selvaraj and Selvakumaran [12] and for $p=1$, was introduced by Aouf and Mostafa [1].

Remarks 1. (*i*) Taking $m=0$ and $b_k = \dfrac{(\alpha_1)_{k-1} ... (\alpha_q)_{k-1}}{(\beta_1)_{k-1} ... (\beta_s)_{k-1} (1)_{k-1}}$ $(\alpha_i, \beta_j \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}, \; (i=1,2,...q), \; (j=1,2,...s), q \leq s+1, q, s \in \mathbb{N}_0$ in (1.4), the operator $D^m_{\lambda,p}(f*g)$ reduces to the Dziok-Srivastava operator $H_{p,q,s}(\alpha_1)$ which generalizes many other operators (see [6]);

(*ii*) Taking $m=0$ and $b_k = \frac{p+l+\lambda(k-p)}{p+l}$ $(\lambda > 0; p \in \mathbb{N}; l, n \in \mathbb{N}_0)$ in (1.4), the operator $D^m_{\lambda,p}(f*g)$ reduces to Catas operator $I^n_p(l,\lambda)$ which generalizes many other operators (see [5]).

Definition 1. A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{N}^m_{p,\lambda}(g, \alpha, \delta, A, B)$ if it satisfies the following subordination condition:

$$(1+\alpha) \left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)} \right)^{\delta} - \alpha \frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)} \left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)} \right)^{\delta} \prec \frac{1+Az}{1+Bz} \tag{1.6}$$

$$(g \in A(p); \lambda > 0; 0 < \delta < 1; \; \alpha \in \mathbb{C}; \; -1 \leq B \leq 1, \; A \neq B, A \in \mathbb{R}; p \in \mathbb{N}, m \in \mathbb{N}_0; z \in U),$$

where all the powers are principal values. Furthermore, the function $f \in \mathcal{N}^m_{p,\lambda}(g, \alpha, \delta, \beta)$ if and only if $f, g \in \mathcal{A}(p)$ and

$$\Re \left\{ (1+\alpha) \left(\tfrac{z^p}{D^m_{\lambda,p}(f*g)(z)} \right)^{\delta} - \alpha \tfrac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)} \left(\tfrac{z^p}{D^m_{\lambda,p}(f*g)(z)} \right)^{\delta} \right\} > \beta \quad (0 \leq \beta < 1; z \in U),$$

we write $\mathcal{N}^m_{p,\lambda}(g, 0; \delta; \beta) = \mathcal{N}^m_{p,\lambda}(g, \delta; \beta)$. We note that:

(*i*) $\mathcal{N}^0_{1,1}\left(\frac{z}{1-z}, \alpha; \delta; A, B \right) = \mathcal{N}(\alpha, \delta; A, B)$, where $\mathcal{N}(\alpha, \delta; A, B)$ is the class defined by Wang el. at [15];

(*ii*) $\mathcal{N}^1_1\left(\frac{z}{1-z}, -1, \delta; 1-2\beta, -1 \right) = \mathcal{N}(\delta; \beta)$ $(0 \leq \beta < 1)$, where $\mathcal{N}(\delta; \beta)$ is the class of non-Bazilevič functions of order β which were considered by Tuneski and Darus [14];

(*iii*) $\mathcal{N}^0_{1,1}\left(\frac{z}{1-z}, -1, \delta; 1, -1 \right) = \mathcal{N}(\delta)$, where $\mathcal{N}(\delta)$ is the class of non-Bazilevič functions which introduced by Obradovic [10];

(iv) $\mathcal{N}_{p,1}^{0}\left(z^p+\sum\limits_{k=p+1}^{\infty}\frac{(\mu+p)_k(c)_k}{(a)_k(1)_k}z^k,\alpha;\delta;A,B\right)=\mathcal{N}_{p,\mu}^{\alpha,\delta}(a,c;A,B)$ $(a,c\in R\backslash Z_0^-,\mu>-p)$, where $\mathcal{N}_{p,\mu}^{\alpha,\delta}(a,c;A,B)$ is the class defined by Wang et al. [16];

(v) $\mathcal{N}_{p,1}^{0}\left(\frac{z^p}{1-z},\alpha;\delta;A,B\right)=\mathcal{N}_p(\alpha;\delta;A,B)$, where $\mathcal{N}_p(\alpha,\delta;A,B)$ is the class of non-Bazilevic functions defined by Aouf and Seoudy [2, with $n=1$].

In the present paper, we prove some subordination and superordination properties, convolution results, distortion theorems and inequality properties for the class $\mathcal{N}_{p,\lambda}^{m}(g,\alpha,\delta,A,B)$.

2. Definitions and Preliminaries

In order to establish our main results, we need the following definition and lemmas.

Definition 2 [8]. *Denote by $\mathcal{Q}$ the set of all functions f that are analytic and injective on $\bar{U}\backslash E(f)$, where*

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and such that $f'(\zeta)\neq0$ for $\zeta\in\bar{U}\backslash E(f)$.

Lemma 1 [9]. *Let the function h be analytic and convex (univalent) in U with $h(0)=1$. Suppose also that the function $p(z)$ given by*

$$p(z)=1+c_nz^n+c_{n+1}z^{n+1}+... \tag{2.1}$$

is analytic in U. If

$$p(z)+\frac{zp'(z)}{\gamma}\prec h(z)\qquad(\Re(\gamma)\geq\mathbf{0};\gamma\neq0;\ z\in U), \tag{2.2}$$

then

$$p(z)\prec q(z)=\frac{\gamma}{n}z^{-\frac{\gamma}{n}}\int_0^z t^{\frac{\gamma}{n}-1}\,h(t)\,dt\prec h(z),$$

and $q(z)$ is the best dominant.

Lemma 2 [13]. *Let q be a convex univalent function in U and $\sigma\in\mathbb{C},\eta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ such that:*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function p is analytic in U and

$$\sigma p(z) + \eta z p^{'}(z) \prec \sigma q(z) + \eta z q^{'}(z),$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 3 [8]. *Let* q *be convex univalent in* U *and* $\varsigma \in \mathbb{C}$. *Further assume that* $\Re(\varsigma) > 0$. *If*

$$p(z) \in H[q(0), 1] \cap \mathcal{Q},$$

and

$$p(z) + \varsigma z p^{'}(z)$$

is univalent in U, *then*

$$q(z) + \varsigma z q^{'}(z) \prec p(z) + \varsigma z p^{'}(z),$$

implies $q(z) \prec p(z)$ *and* q *is the best subordinant.*

Lemma 4 [7]. *Let* $\mathcal{F}$ *be analytic and convex in* U . *If*

$$f, g \in \mathcal{A} \quad and \quad f, g \prec \mathcal{F}$$

then

$$\lambda f + (1-\lambda) g \prec \mathcal{F} \qquad (0 \le \lambda \le 1).$$

Lemma 5 [11]. *Let*

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$$

be analytic in U *and*

$$g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$$

be analytic and convex in U. *If* $f \prec g$, *then*

$$|a_k| < |b_1| \qquad (k \in \mathbb{N}).$$

3.Main results

Unless otherwise mentioned, we assume throughout this paper that $g \in A(p), p \in \mathbb{N}, m \in \mathbb{N}_0, \lambda > 0, 0 < \delta < 1,$ $\alpha \in \mathbb{C}$, $-1 \le B \le 1$, $A \ne B$, $A \in \mathbb{R}$ and $z \in U$.

Theorem 1. *Let* $f \in \mathcal{N}_{p,\lambda}^{m}(g, \alpha, \delta, A, B)$ *with* $\Re(\alpha) > 0$. *Then*

$$\left(\frac{z^p}{D_{\lambda,p}^m (f*g)(z)}\right)^{\delta} \prec q(z) = \frac{p\delta}{n\alpha\lambda} \int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1} \frac{1+Azu}{1+Bzu} du \prec \frac{1+Az}{1+Bz} \qquad (3.1)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z)=\left(\frac{z^p}{D^m_{p,\lambda}(f*g)(z)}\right)^{\delta}\quad (z\in U). \tag{3.2}$$

Then the function $p(z)$ is of the form (2.1) and analytic in U. By taking logarithmic differentiation of the both sides of (3.2) with respect to z, we have

$$p(z)+\frac{\alpha\lambda}{p\delta}zp'(z)=(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}-\alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}. \tag{3.3}$$

Since $f\in\mathcal{N}^m_{p,\lambda}(g,\alpha,\delta,A,B)$, we have

$$p(z)+\frac{\alpha\lambda}{p\delta}zp'(z)\prec\frac{1+Az}{1+Bz}.$$

Applying Lemma 1 to (3.3) with $\gamma=\frac{p\delta}{\alpha\lambda}$, we get

$$\begin{aligned}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} &\prec q(z)=\frac{p\delta}{n\alpha\lambda}z^{-\frac{p\delta}{n\alpha\lambda}}\int_0^z t^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+At}{1+Bt}\,dt\\ &=\frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+Azu}{1+Bzu}du\prec\frac{1+Az}{1+Bz}\end{aligned} \tag{3.4}$$

and $q(z)$ is the best dominant. The proof of Theorem 1 is thus completed.

Theorem 2. *Let $q(z)$ be univalent in U, $\alpha\in\mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{p\delta}{\alpha\lambda}\right)\right\}. \tag{3.5}$$

If $f\in\mathcal{A}(p)$ satisfies the following subordination condition:

$$(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}-\alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}\prec q(z)+\frac{\alpha\lambda}{p\delta}zq'(z), \tag{3.6}$$

then

$$\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}\prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ be defined by (3.2). We know that (3.3) holds true. Combining (3.3) and (3.6), we find that

$$p(z) + \frac{\alpha\lambda}{p\delta} z p'(z) \prec q(z) + \frac{\alpha\lambda}{p\delta} z q'(z). \tag{3.7}$$

By using Lemma 2 and (3.7), we easily get the assertion of Theorem 2.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we get the following result.

Corollary 1. *Let $\alpha \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$. Suppose also that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\Re\left(\frac{p\delta}{\alpha\lambda}\right)\right\}.$$

If $f \in \mathcal{A}(p)$ satisfies the following subordination condition:

$$(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta - \alpha \frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta \prec \frac{1+Az}{1+Bz} + \frac{\alpha\lambda}{p\delta}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $A = 1$ and $B = -1$ in Corollary 1, we get the following result.

Corollary 2. *Let $\alpha \in \mathbb{C}^*$ and suppose also that*

$$\Re\left(\frac{1+z}{1-z}\right) > \max\left\{0, -\Re\left(\frac{p\delta}{\alpha\lambda}\right)\right\}.$$

If $f \in \mathcal{A}(p)$ satisfies the following subordination:

$$(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta - \alpha \frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta \prec \frac{1+z}{1-z} + \frac{\alpha\lambda}{p\delta}\frac{2z}{(1-z)^2},$$

then

$$\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta \prec \frac{1+z}{1-z}$$

and the function $\frac{1+z}{1-z}$ is the best dominant.

We now derive the following superordination result.

Theorem 3. *Let q be convex univalent in U, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Also let*

$$\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta \in H[q(0), 1] \cap \mathcal{Q}$$

and

$$(1+\alpha)\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta-\alpha\frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^m(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta$$

be univalent in U*. If* $f\in\mathcal{A}(p)$ *satisfies the following superordination:*

$$q(z)+\frac{\alpha\lambda}{p\delta}zq'(z)\prec(1+\alpha)\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta-\alpha\frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^m(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta,$$

then

$$q(z)\prec\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta$$

and the function $q(z)$ *is the best subordinant.*

Proof. Let the function $p(z)$ be defined by (3.2). Then

$$\begin{aligned} q(z)+\frac{\alpha\lambda}{p\delta}zq'(z) &\prec (1+\alpha)\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta-\alpha\frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^m(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta \\ &= p(z)+\frac{\alpha\lambda}{p\delta}zp'(z). \end{aligned}$$

An application of Lemma 3 yields the assertion of Theorem 3.

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 3, we get the following corollary.

Corollary 3. *Let* $-1\le B<A\le 1$*,* $\alpha\in\mathbb{C}$ *with* $\Re(\alpha)>0$*. Also let*

$$\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta\in H[q(0),1]\cap\mathcal{Q}$$

and

$$(1+\alpha)\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta-\alpha\frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^m(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta$$

be univalent in U*. If* $f\in\mathcal{A}(p)$ *satisfies the following superordination condition:*

$$\frac{1+Az}{1+Bz}+\frac{\alpha\lambda}{p\delta}\frac{(A-B)z}{(1+Bz)^2}\prec(1+\alpha)\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta-\alpha\frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^m(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta,$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z^p}{D_{\lambda,p}^m(f*g)(z)}\right)^\delta$$

and the function $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Combining Theorems 2 and 3, we easily get the following Sandwich result.

Theorem 4. *Let q_1 be convex univalent and let q_2 be univalent in U, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Let q_2 satisfies (3.5). If*

$$\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} \in H\left[q_1(0), 1\right] \cap \mathcal{Q}$$

and

$$(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} - \alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}$$

be univalent in U, also

$$\begin{aligned} q_1(z) + \frac{\alpha\lambda}{p\delta} z q_1'(z) &\prec (1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} - \alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} \\ &\prec q_2(z) + \frac{\alpha\lambda}{p\delta} z q_2'(z), \end{aligned}$$

then

$$q_1(z) \prec \left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Theorem 5. *If $\alpha > 0$ and $f \in \mathcal{N}^m_{p,\lambda}(g,\delta;\beta)$ $(0 \le \beta < 1)$. Then $f \in \mathcal{N}^m_{p,\lambda}(g,\alpha,\delta,\beta)$ for $|z| < R$, where*

$$R = \left(\sqrt{\left(\frac{n\alpha\lambda}{p\delta}\right)^2 + 1} - \frac{n\alpha\lambda}{p\delta}\right)^{\frac{1}{n}}. \tag{3.8}$$

The bound R is the best possible.

Proof. We begin by writing

$$\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} = \beta + (1-\beta)\,p(z) \qquad (z \in U). \tag{3.9}$$

Then clearly, the function $p(z)$ is of the form (2.1), analytic and has a positive real part in U. By taking the derivatives of both sides of (3.9), we get

$$\begin{aligned} &\frac{1}{1-\beta}\left\{(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} - \alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} - \beta\right\} \\ &= p(z) + \frac{\alpha\lambda}{p\delta} z p'(z). \end{aligned} \tag{3.10}$$

By making use of the following well-known estimate (see [3, Theorem 1]):

$$\frac{\left|zp'(z)\right|}{\Re\{p(z)\}} \leq \frac{2nr^n}{1-r^{2n}} \qquad (|z| = r < 1)$$

in (3.10), we obtain

$$\begin{aligned} &\Re\left(\frac{1}{1-\beta}\left\{(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta - \alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta - \beta\right\}\right) \\ \geq\ &\Re\{p(z)\}\left(1-\frac{2\alpha\lambda nr^n}{p\delta(1-r^{2n})}\right). \end{aligned} \tag{3.11}$$

It is seen that the right-hand side of (3.11) is positive, provided that $r < R$, where R is given by (3.8). In order to show that the bound R is the best possible, we consider the function $f \in \mathcal{A}(p)$ defined by

$$\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta = \beta + (1-\beta)\frac{1+z^n}{1-z^n} \qquad (z \in U).$$

By noting that

$$\begin{aligned} &\frac{1}{1-\beta}\left\{(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta - \alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta - \beta\right\} \\ =\ &\frac{1+z^n}{1-z^n} + \frac{2\alpha\lambda nz^n}{p\delta(1-z^n)^2} = 0, \end{aligned} \tag{3.12}$$

for $z = R\ \exp\left(\frac{\pi i}{n}\right)$, we conclude that the bound is the best possible. Theorem 5 is thus proved.

Theorem 6. *Let* $\alpha_2 \geq \alpha_1 \geq 0$ *and* $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. *Then*

$$\mathcal{N}^m_{p,\lambda}(g, \alpha_2, \delta; A_2, B_2) \subset \mathcal{N}^m_{p,\lambda}(g, \alpha_1, \delta; A_1, B_1). \tag{3.13}$$

Proof. Let $f \in \mathcal{N}^m_{p,\lambda}(g, \alpha_2, \delta; A_2, B_2)$. Then we have

$$(1+\alpha_2)\left(\frac{z^p}{D^m_{p,\lambda}(f*g)(z)}\right)^\delta - \alpha_2\frac{D^{m+1}_{p,\lambda}(f*g)(z)}{D^m_{p,\lambda}(f*g)(z)}\left(\frac{z^p}{D^m_{p,\lambda}(f*g)(z)}\right)^\delta \prec \frac{1+A_2z}{1+B_2z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$(1+\alpha_2)\left(\frac{z^p}{D^m_{p,\lambda}(f*g)(z)}\right)^\delta - \alpha_2\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^\delta \prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \tag{3.14}$$

that is $f \in \mathcal{N}_{p,\lambda}^{m}(g, \alpha_2, \delta; A_1, B_1)$. Thus the assertion of Theorem 6 holds for $\alpha_2 = \alpha_1 \geq 0$. If $\alpha_2 > \alpha_1 \geq 0$, by Theorem 1 and (3.14), we know that $f \in \mathcal{N}_{p,\lambda}^{m}(g, 0, \delta; A_1, B_1)$, that is,

$$\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} \prec \frac{1+A_1 z}{1+B_1 z}, \tag{3.15}$$

At the same time, we have

$$\begin{aligned} &(1+\alpha_1)\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} - \alpha_1 \frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^{m}(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} \\ = \; &\left(1-\frac{\alpha_1}{\alpha_2}\right)\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} + \frac{\alpha_1}{\alpha_2}\left[(1+\alpha_2)\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta}\right. \\ &\left. -\alpha_2 \frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^{m}(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta}\right]. \end{aligned} \tag{3.16}$$

Moreover, since $0 \leq \frac{\alpha_1}{\alpha_2} < 1$,and the function $\frac{1+A_1 z}{1+B_1 z}$ $(-1 \leq B_1 < A_1 \leq 1)$ is analytic and convex in U. Combining $(3.14) - (3.16)$ and Lemma 4, we find that

$$(1+\alpha_1)\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} - \alpha_1 \frac{D_{\lambda,p}^{m+1}(f*g)(z)}{D_{\lambda,p}^{m}(f*g)(z)}\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} \prec \frac{1+A_1 z}{1+B_1 z},$$

that is $f \in \mathcal{N}_{p,\lambda}^{m}(g, \alpha_1, \delta; A_1, B_1)$, which implies that the assertion (3.13) of Theorem 6 holds.

Theorem 7. *Let* $f \in \mathcal{N}_{p,\lambda}^{m}(g, \alpha, \delta; A, B)$ *with* $\alpha > 0$ *and* $-1 \leq B < A \leq 1$. *Then*

$$\frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1-Au}{1-Bu}\,du < \Re\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} < \frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+Au}{1+Bu}\,du. \tag{3.17}$$

The extremal function of (3.17) *is defined by*

$$F(z) = D_{\lambda,p}^{m}(f*g)(z) = z^p\left(\frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+Auz^n}{1+Buz^n}\,du\right)^{-\frac{1}{\delta}}. \tag{3.18}$$

Proof. Let $f \in \mathcal{N}_{p,\lambda}^{m}(g, \alpha, \delta; A, B)$ with $\alpha > 0$. From Theorem 1, we know that (3.1) holds, which implies that

$$\begin{aligned} \Re\left(\frac{z^p}{D_{\lambda,p}^{m}(f*g)(z)}\right)^{\delta} &< \sup_{z\in U}\Re\left\{\frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+Azu}{1+Bzu}\,du\right\} \\ &\leq \frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\sup_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)du \\ &< \frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+Au}{1+Bu}\,du \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\Re\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta} &> \inf_{z\in U}\Re\left\{\frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+Azu}{1+Bzu}\,du\right\}\\ &\geq \frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\inf_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)du\\ &> \frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1-Au}{1-Bu}\,du. \qquad (3.20)\end{aligned}$$

Combining (3.19) and (3.20), we get (3.17). By noting that the function $F(z)$ defined by (3.18) belongs to the class $\mathcal{N}^m_{p,\lambda}(g,\alpha,\delta;A,B)$, we obtain that equality (3.17) is sharp. The proof of Theorem 7 is evidently completed.

By similarly applying the method of proof of Theorem 7, we easily get the following result.

Corollary 4. *Let $f\in\mathcal{N}^m_{p,\lambda}(g,\alpha,\delta;A,B)$ with $\boldsymbol{\alpha>0}$ and $-1\leq B<A\leq 1$. Then*

$$\frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1+Au}{1+Bu}\,du<\Re\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}<\frac{p\delta}{n\alpha\lambda}\int_0^1 u^{\frac{p\delta}{n\alpha\lambda}-1}\frac{1-Au}{1-Bu}\,du.$$

The extremal function of (3.21) *is defined by* (3.18).

Theorem 8. *Let*

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k\in\mathcal{N}^m_{p,\lambda}(g,\alpha,\delta;A,B). \qquad (3.21)$$

Then

$$|a_{p+1}|\leq p\left(\frac{p+\lambda}{p}\right)^{-m}\frac{(A-B)}{|p\delta+\alpha\lambda|\,|b_{p+1}|}. \qquad (3.22)$$

The inequality (3.22) *is sharp, with the extremal function defined by* (3.18).

Proof. Combining (1.6) and (3.21), we obtain

$$\begin{aligned}&(1+\alpha)\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}-\alpha\frac{D^{m+1}_{\lambda,p}(f*g)(z)}{D^m_{\lambda,p}(f*g)(z)}\left(\frac{z^p}{D^m_{\lambda,p}(f*g)(z)}\right)^{\delta}\\ =&\ 1+\left(\frac{p+\lambda}{p}\right)^m\left(\delta+\frac{\alpha\lambda}{p}\right)a_{p+1}b_{p+1}z+...\prec\frac{1+Az}{1+Bz}. \qquad (2.23)\end{aligned}$$

An application of Lemma 5 to (3.24) yields

$$\left|\left(\frac{p+\lambda}{p}\right)^m\left(\delta+\frac{\alpha\lambda}{p}\right)a_{p+1}b_{p+1}\right|<A-B. \qquad (3.24)$$

Thus, from (3.24), we easily arrive at (3.22) asserted by Theorem 8.

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