Acta Universitatis Apulensis
ISSN: 1582-5329

No. 31/2012
pp. 163-177

# ON A CLASS OF ANALYTIC FUNCTION DEFINED USING DIFFERENTIAL OPERATOR

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Abstract. In this paper we introduce a new class of analytic functions of complex order involving a family of generalized differential operators and we discuss the sufficient conditions, estimation of coefficients and certain subordination results. Using this one can derive numerous known results as special cases.

2000 *Mathematics Subject Classification*: 30C45.

## 1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0 \tag{1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{P}$ be the class of functions $f(z)$ in $\mathcal{A}$ which are univalent in $U$. The Hadamard product of two functions $f(z)$ given by (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is defined as

$$(f * g)(z) = (g * f)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let $f(z)$ and $g(z)$ be analytic in the unit disc $U$. Then $f(z)$ is said to be subordinate to $g(z)$ in $U$, if there exists a Schwartz function $w(z)$, analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$. Further if $g(z)$ is univalent if $f(0) = g(0)$ and if $f(U) \subset g(U)$, then we write $f \prec g$.

For complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s; (\beta_j \in \mathbb{C} \setminus \mathcal{Z}_0^-; \mathcal{Z}_0^- = \{0, -1, -2, \ldots\}$ for $j = 1, 2, .., s)$, we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z)$ as

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_q)_k z^k}{(\beta_1)_k(\beta_2)_k \ldots (\beta_s)_k k!},$$

$$(q \leq s+1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U),$$

where $\mathbb{N}$ denotes the set of all positive integers and $(x)_k$ is the Pochhammer symbol defined in terms of gamma functions, as

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{if} \quad k=0 \\ x(x+1)\dots(x+k-1) & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

Corresponding to the function $g_{q,s}(\alpha_1, \beta_1; z)$ defined by

$$g_{q,s}(\alpha_1, \beta_1; z) = z\, {}_qF_s(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s; z).$$

Recently in [9, 14] an operator $\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) : \mathcal{A} \to \mathcal{A}$ is defined by

$$\begin{aligned} \mathcal{D}^0_{\lambda,\mu}(\alpha_1, \beta_1)f(z) &= f(z) * g_{q,s}(\alpha_1, \beta_1; z), \\ \mathcal{D}^1_{\lambda,\mu}(\alpha_1, \beta_1)f(z) &= (1-\lambda+\mu)(f(z) * g_{q,s}(\alpha_1, \beta_1; z)) + (\lambda-\mu)z(f(z) * g_{q,s}(\alpha_1, \beta_1; z))' \\ &\quad + \lambda\mu z^2(f(z) * g_{q,s}(\alpha_1, \beta_1; z)'', \\ \mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) &= \mathcal{D}^1_{\lambda,\mu}(\mathcal{D}^{m-1}_{\lambda,\mu}(\alpha_1, \beta_1)f(z)), \end{aligned}$$

where $0 \leq \mu \leq \lambda \leq 1$ and $m \in \mathbb{N}_0$. By the above definition, it is easy to note that

$$\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) = z + \sum_{k=2}^{\infty} [1+(k-1)(\lambda-\mu+k\mu\lambda)]^m \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1}\dots(\beta_s)_{k-1}(k-1)!} a_k z^k.$$

For brevity, let us take

$$B_k = \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1}\dots(\beta_s)_{k-1}(k-1)!}.$$

Hence we have

$$\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) = z + \sum_{k=2}^{\infty} [1+(k-1)(\lambda-\mu+k\mu\lambda)]^m B_k a_k z^k.$$

For suitable values of $\alpha_{i's}, \beta_{j's}, q, s, \lambda$ and $\mu$ we can deduce several operators as a special case of this operator. For example see [1, 5, 12].

Using this operator $\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)$, we define a class $M$ of functions $f \in \mathcal{A}$ which satisfies the inequality

$$1 + \frac{1}{b}\left(\frac{\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)} - 1\right) \prec \frac{1+Az}{1+Bz}, \tag{2}$$

for $z \in U$, $b \in \mathbb{C} \setminus \{0\}$ and $A$ and $B$ are arbitrary fixed numbers such that $-1 \leq B \leq A \leq 1$.

We note that by specializing $b$, $m$, $\lambda$, $q, s$, $\alpha_{i's}$, $\beta_{i's}$, $A$, and $B$ in the function class $M$, we obtain several well-known as well as new subclasses of analytic functions. Here we list a few of them:

1. If we let $\lambda = 1$, $\mu = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, then the class $M$ reduces to the well- known class
$$\mathcal{H}^m(b; A, B) := \left\{ f : \, f \in \mathcal{A}, \, 1 + \frac{1}{b}\left( \frac{\mathscr{D}^{m+1} f(z)}{\mathscr{D}^m f(z)} - 1 \right) \prec \frac{1+Az}{1+Bz}, \; z \in \mathcal{U} \right\}$$
where $\mathscr{D}^m f$ is the well- known Sălăgean operator. The class $\mathcal{H}^m(\delta; A, B)$ has been introduced and studied by Attiya in [4].

2. For a choice of the parameter $\lambda = 1$, $\mu = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $A = 1$ and $B = -K$, the class $M$ reduces to the class
$$\mathcal{H}^m(b; K) := \left\{ f : \, f \in \mathcal{A}, \, \left| \frac{b - 1 + \frac{\mathscr{D}^{m+1} f(z)}{\mathscr{D}^m f(z)}}{b} - K \right| < K, \; z \in \mathcal{U} \right\}$$
where $K > \frac{1}{2}$. The class $\mathcal{H}^m(b; K)$ has been introduced and studied by Aouf, Darwish and Attiya in [3].

3. If we take $\lambda = 1$, $\mu = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $b = 1 - \alpha$ $(0 \leq \alpha < 1)$, $A = 1$ and $B = -1$ then the class $M$ reduces to the class
$$\mathcal{S}^*_m(\alpha) := \left\{ f : \, f \in \mathcal{A}, \, Re\left\{ \frac{\mathscr{D}^{m+1} f(z)}{\mathscr{D}^m f(z)} \right\} > \alpha, \, z \in \mathcal{U} \right\}.$$
The class $\mathcal{S}^*_m(\alpha)$ has been introduced and studied by E. Kadioğlu in [8].

Apart from the these, several other well known as well as new classes of analytic functions can be obtained by specializing the parameters involved in the class $M$. For example, see [2, 3, 10, 11, 13, 15, 16].

Let $\Omega$ denote the class of bounded analytic functions $w(z)$ in $U$ which satisfy the condition $w(0) = 1$ and $|w(z)| < 1$ for $z \in U$.

## 2. A sufficient condition for a function to be in $M$

**Theorem 1.** *Let the function $f(z)$ be defined by (1) and let*

$$\sum_{k=2}^{\infty} [1 + (k-1)(\lambda - \mu + k\mu\lambda)]^m \{(k-1)(\lambda - \mu + k\mu\lambda) + |(A-B)b - B(k-1)(\lambda - \mu + k\mu\lambda)|\} B_k |a_k| \leq (A-B)|b| \tag{3}$$

*hold, then $f(z)$ belongs to $M$.*

*Proof.* Suppose that the inequality holds, then we have for $z \in U$,

$$\left|\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - \mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)\right| - \left|(A-B)b\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - B[\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - \mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]\right|$$

$$= \left|\sum_{k=2}^{\infty}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m\left[(k-1)(\lambda-\mu+k\mu\lambda)\right]B_k a_k z^k\right|$$

$$-\left|(A-B)bz + \sum_{k=2}^{\infty}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m \left[(A-B)b - B(k-1)(\lambda-\mu+k\mu\lambda)\right]B_k a_k z^k\right|$$

$$\leq \sum_{k=2}^{\infty}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m\{(k-1)(\lambda-\mu+k\mu\lambda) + |(A-B)b - B(k-1)(\lambda-\mu+k\mu\lambda)|\}B_k|a_k|r^k - (A-B)|b|r.$$

Letting $r \to 1^-$, we have

$$|\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - \mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)| - |(A-B)b\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - B[\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - \mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)]|$$

$$\leq \sum_{k=2}^{\infty}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m\{(k-1)(\lambda-\mu+k\mu\lambda) + |(A-B)b - B(k-1)(\lambda-\mu+k\mu\lambda)|\}B_k|a_k|r^k - (A-B)|b|r \leq 0.$$

Hence it follows that

$$\frac{\left|\frac{\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)} - 1\right|}{\left|B\left[\frac{\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)} - 1\right] - (A-B)b\right|} < 1.$$

Letting

$$w(z) = \frac{\frac{\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)} - 1}{B\left[\frac{\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)} - 1\right] - (A-B)b},$$

then $w(0) = 0$, $w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Hence we have

$$\frac{\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)}{\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)} = \frac{1+[B+b(A-B)]w(z)}{1+Bw(z)}$$

which shows $f(z) \in M$.

If we let $\lambda = 1$, $\mu = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in Theorem 1 , we have the following result.

**Corollary 1.** *Let $f \in \mathcal{A}$ and let*

$$\sum_{k=2}^{\infty} k^m \big\{(k-1)+ \mid (A-B)b - B(k-1) \mid \big\} \mid a_n \mid \leq (A-B) \mid b \mid \tag{4}$$

*holds, then $f(z)$ belongs to $\mathcal{H}^m(\delta; A, B)$.*

If we let $\lambda = 1$, $\mu = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $A = 1$ and $B = -K$ in Theorem 1, we get the following interesting result.

**Corollary 2.** [3] *Let the function $f(z)$ defined by (1) and let*

$$\sum_{k=2}^{\infty} \{(k-1)+ \mid b(1+u) + u(k-1) \mid\} \, k^m \mid a_k \mid \leq |b(1+u)| \tag{5}$$

*holds, then $f(z)$ belongs to $\mathcal{H}^m(b; K)$, where $u = 1 - \dfrac{1}{K}$ $\left(K > \dfrac{1}{2}\right)$.*

## 3. Estimation of Coefficients

**Theorem 2.** *Let the function $f(z)$ defined by (1) be in the class $M$.*
*(a)If $(A-B)^2|b|^2 > [2(A-B)B\Re b + (1-B^2)(k-1)(\lambda-\mu+\lambda k\mu)](k-1)(\lambda-\mu+\lambda k\mu)$, let*

$$G = \frac{(A-B)^2|b|^2}{[2(A-B)B\Re\{b\} + (1-B^2)(k-1)(\lambda-\mu+\lambda k\mu)](k-1)(\lambda-\mu+\lambda k\mu)}$$

*where $k = 2, 3, \ldots, m-1$. Let $N = \lfloor G \rfloor$ (Gauss symbol), the greatest integer not greater than $G$, then*

$$|a_j| \leq \frac{\prod_{k=2}^{j} |(A-B)b - (k-2)B|}{[1+(j-1)(\lambda-\mu+j\mu\lambda)]^m (\lambda-\mu+j\mu\lambda)^{j-1}(j-1)! B_j} \tag{6}$$

*for* $j = 2, 3, \ldots, N+2$ *and*

$$|a_j| \leq \frac{\prod_{k=2}^{N+3} |(A-B)b-(k-2)B|}{[1+(j-1)(\lambda-\mu+j\mu\lambda)]^m(\lambda-\mu+j\mu\lambda)^{j-1}(j-1)(N+1)!B_j} \tag{7}$$

*for* $j > N+2$.
*(b) If* $(A-B)^2|b|^2 \leq [2(A-B)B\Re b+(1-B^2)(k-1)(\lambda-\mu+\lambda k\mu)](k-1)(\lambda-\mu+\lambda k\mu)$*,then*

$$|a_j| \leq \frac{(A-B)|b|}{[1+(j-1)(\lambda-\mu+j\mu\lambda)]^m(\lambda-\mu+j\mu\lambda)(j-1)B_j} \tag{8}$$

*for* $j \geq 2$. *The bounds (6) and (8)are sharp for all admissible* $A, B, b \in \mathbb{C}\setminus\{0\}$ *and for each* $j$.

*Proof.* Since $f(z) \in M$, the inequality (2)gives

$$\begin{aligned} &|\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - \mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)| = \\ &\quad \{[(A-B)b+B]\mathcal{D}^{m}_{\lambda,\mu}(\alpha_1,\beta_1)f(z) - B[\mathcal{D}^{m+1}_{\lambda,\mu}(\alpha_1,\beta_1)f(z)\}w(z). \end{aligned} \tag{9}$$

Equation (9) may be rewritten as

$$\sum_{k+2}^{\infty}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m(k-1)(\lambda-\mu+\lambda k\mu)B_k a_k z^k$$

$$= \{(A-B)bz+\sum_{k=2}^{\infty}[(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)][1+(k-1)(\lambda-\mu+k\mu\lambda)]^m B_k a_k z^k\}w(z).$$

Or equivalently,

$$\sum_{k=2}^{j}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m(k-1)(\lambda-\mu+k\mu\lambda)B_k a_k z^k + \sum_{k=j+1}^{\infty} c_k z^k$$

$$= \{(A-B)bz+\sum_{k=2}^{j-1}[(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)][1+(k-1)(\lambda-\mu+k\mu\lambda)]^m B_k a_k z^k\}w(z)$$

for certain coefficients $c_k$. Since $|w(z)| < 1$, we have

$$\left|\sum_{k=2}^{j}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^m(k-1)(\lambda-\mu+k\mu\lambda)B_k a_k z^k + \sum_{k=j+1}^{\infty} c_k z^k\right|$$

$$\leq \left|(A-B)bz+\sum_{k=2}^{j-1}[(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)] \right.$$
$$\left. [1+(k-1)(\lambda-\mu+k\mu\lambda)]^m B_k a_k z^k \right|.$$

Let $z = re^{i\theta}, r < 1$. Applying the Parseval's formula on both sides of the above inequality and a simple computation we get

$$\sum_{k=2}^{j}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^{2m}(k-1)^2(\lambda-\mu+k\mu\lambda)^2 B_k^2|a_k|^2r^{2k}+\sum_{k=j+1}^{\infty}|c_k|^2r^{2k}|$$

$$\leq (A-B)^2|b|^2r^2+\sum_{k=2}^{j-1}|(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)|^2[1+(k-1)(\lambda-\mu+k\mu\lambda)]^{2m}B_k^2a_k^2\mu^{2k}.$$

Let $r \to 1^-$. Then on simplification we get

$$[1+(j-1)(\lambda-\mu+j\mu\lambda)]^{2m}(j-1)^2(\lambda-\mu+j\mu\lambda)^2B_j^2|a_j|^2 \tag{10}$$

$$\leq (A-B)^2|b|^2+\sum_{k=2}^{j-1}\{|(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)|^2-(k-1)^2(\lambda-\mu+k\mu\lambda)^2\}$$
$$\times[1+(k-1)(\lambda-\mu+k\mu\lambda)]^{2m}B_k^2|a_k|^2$$

for $j \geq 2$.

Now the following two cases arise

(a) $(A-B)^2|b|^2 > [2(A-B)B\Re b+(1-B^2)(k-1)(\lambda-\mu+k\mu\lambda)](k-1)(\lambda-\mu+k\mu\lambda)$ suppose that $j \leq N+2$. Then

$$|a_2| \leq \frac{(A-B)|b|}{(1+\lambda-\mu+2\mu\lambda)(\lambda-\mu+2\mu\lambda)B_2}$$

which gives (6) for $j=2$. We establish (6) for $j < N+2$ from (10) by mathematical induction. Suppose (6) is valid for $j = 2, 3, \ldots, (k-1)$. Then it follows from (10) that

$$[1+(j-1)(\lambda-\mu+j\mu\lambda)]^{2m}(j-1)^2(\lambda-\mu+j\mu\lambda)^2B_j^2|a_j|^2$$

$$\leq (A-B)^2|b|^2+\sum_{k=2}^{j-1}\{|(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)|^2-(k-1)^2(\lambda-\mu+k\mu\lambda)^2\}$$

$$\times[1+(k-1)(\lambda-\mu+k\mu\lambda)]^{2m}B_k^2|a_k|^2$$

$$\leq (A-B)^2|b|^2+\sum_{k=2}^{j-1}\{|(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)|^2-(k-1)^2(\lambda-\mu+k\mu\lambda)^2\}$$

$$\times[1+(k-1)(\lambda-\mu+k\mu\lambda)]^{2m}B_k^2$$

$$\times\frac{\prod_{n=2}^{k}|(A-B)b-(n-2)B|^2}{[1+(k-1)(\lambda-\mu+k\mu\lambda)]^{2m}B_k^2}\{(\lambda-\mu+k\mu\lambda)^{k-1}(k-1)!\}^2$$

$$=(A-B)^2|b|^2+[|(A-B)b-B(\lambda-\mu+2\mu\lambda)|^2-(\lambda-\mu+2\mu\lambda)^2]\frac{(A-B)^2b^2}{(\lambda-\mu+2\mu\lambda)^2(1!)^2}$$

$$+\{|(A-B)b-2B(\lambda-\mu+3\mu\lambda)|^2-4(\lambda-\mu+3\mu\lambda)^2\}$$

$$\frac{1}{(\lambda-\mu+3\mu\lambda)^4(2!)^2}(A-B)^2b^2|(A-B)b-B|^2+\ldots.$$

$$=\frac{\prod_{k=2}^{j}|(A-B)b-(k-2)B|^2}{\{(\lambda-\mu+j\mu\lambda)^{j-2}(j-2)!}.$$

Thus we get

$$|a_j|\leq\frac{\prod_{k=2}^{j}|(A-B)b-(k-2)B|^2}{[1+(j-1)(\lambda-\mu+j\mu\lambda)]^m(j-1)!(\lambda-\mu+j\mu\lambda)^{j-1}B_j}.$$

Next we suppose that $j>N+2$. Then (10) gives that

$$[1+(j-1)(\lambda-\mu+j\mu\lambda)]^{2m}(j-1)^2(\lambda-\mu+j\mu\lambda)^2B_j^2|a_j|^2$$

$$\leq (A-B)^2|b|^2+\sum_{k=2}^{N+2}\{|(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)|^2-(k-1)^2(\lambda-\mu+k\mu\lambda)^2\}$$

$$\times[1+(j-1)(\lambda-\mu+j\mu\lambda)]^{2m}B_k^2|a_k|^2$$

$$+\sum_{k=3}^{j-1}\{|(A-B)b-B(k-1)(\lambda-\mu+k\mu\lambda)|^2-(k-1)^2(\lambda-\mu+k\mu\lambda)^2\}[1+(k-1)(\lambda-\mu+k\mu\lambda)]^{2m}B_k^2|a_k|^2$$

on substituting the upper estimates of $a_2,a_3,\ldots,a_{N+2}$ obtained above and simplifying we get (7).

(b) Let $(A-B)^2|b|^2\leq[2(A-B)B\Re b+(1-B^2)(k-1)(\lambda-\mu+\lambda k\mu)](k-1)(\lambda-\mu+\lambda k\mu)$. It follows from (10) that

$$[1+(j-1)(\lambda-\mu+j\mu\lambda)]^{2m}(j-1)^2(\lambda-\mu+j\mu\lambda)^2B_j^2|a_j|^2\leq(A-B)^2|b|^2$$

which proves (8).

The bounds in (6) are sharp for the functions $f(z)$ given by

$$D_{\lambda}^{m}(\alpha_1,\, \beta_1) f(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)b}{B}} & if\, B \neq 0, \\ z\, \exp(Abz) & if\, B = 0. \end{cases}$$

Also, the bounds in (8) are sharp for the functions $f_k(z)$ given by

$$D_{\lambda}^{m}(\alpha_1,\, \beta_1) f_k(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)b}{B\lambda(k-1)}} & if\, B \neq 0, \\ z\, \exp\left(\frac{Ab}{\lambda(k-1)}\, z^{k-1}\right) & if\, B = 0. \end{cases}$$

We remark here that by specializing the parameters, the above result reduces to various other results obtained by several authors.

If we let $\lambda = 1$, $\mu = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in Theorem 2, we get the result due to Attiya [4].

**Corollary 3.** [4] *Let the function $f(z)$ defined by (1) be in the class $\mathcal{H}^m(\delta; A, B)$.*

*(a) If $(A-B)^2 \mid b \mid^2 > (n-1)\big\{2B(A-B)\, Re\{b\} + (1-B^2)(n-1)\big\}$, let*

$$G = \frac{(A-B)^2 \mid b \mid^2}{(n-1)\big\{2B(A-B)\, Re\{b\} + (1-B^2)(k-1)\big\}},$$
$$(for\ n = 2, 3, \dots, m-1),$$

*$M = [G]$ (Gauss symbol) and $[G]$ is the greatest integer not greater than $G$. Then, for $j = 2, 3, \dots, M+2$*

$$\mid a_j \mid \leq \frac{1}{j^m\, (j-1)!} \prod_{n=2}^{j} \mid (A-B)b - (n-2)B \mid \tag{11}$$

*and for $j > M+2$*

$$\mid a_j \mid \leq \frac{1}{j^m (j-1)\, (M+1)!} \prod_{n=2}^{M+3} \mid (A-B)b - (n-2)B \mid .$$

*(b) If $(A-B)^2 \mid b \mid^2 \leq (n-1)\big\{2B(A-B)\, Re\{b\} + (1-B^2)(n-1)\big\}$, then*

$$\mid a_j \mid \leq \frac{(A-B) \mid b \mid}{(j-1)\, j^m}, \quad j \geq 2. \tag{12}$$

*The bounds in (11) and (12) are sharp for all admissible* $A$, $B$, $b \in \mathbb{C} \setminus \{0\}$ *and for each* $j$.

If we let $\lambda = 1$, $\mu = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, $A = 1$ and $B = -K$ in Theorem 2, we have

**Corollary 4.** [3] *Let the function* $f(z)$ *defined by (1) be in the class* $\mathcal{H}^m(b; K)$.

*(a)* *If* $2u(n-1)Re\,\{b\} > (n-1)^2(1-u) - \mid b \mid^2 (1+u)$, *let*

$$G = \left[ \frac{2u(n-1)\,Re\,(b)}{(n-1)^2(1-u) - |b|^2(1+u)} \right]. \qquad for n = 1,\, 3,\, \ldots,\, j-1.$$

*Then, for* $j = 2, 3, \ldots, G+2$,

$$\mid a_j \mid \leq \frac{1}{j^m\,(j-1)!} \prod_{n=2}^{j} \mid (1+u)b + (n-2)u \mid \tag{13}$$

*and for* $j > G+2$,

$$\mid a_j \mid \leq \frac{1}{j^m(j-1)\,(G+1)!} \prod_{n=2}^{G+3} \mid (1+u)b + (n-2)u \mid .$$

*(b)* *If* $2u(n-1)\,Re\,\{b\} \leq (n-1)^2(1-u) - |b|^2(1+u)$, *then*

$$\mid a_j \mid \leq \frac{(1+u) \mid b \mid}{(j-1)\,j^m} \quad j \geq 2. \tag{14}$$

*where* $u = 1 - \dfrac{1}{K}$ *and* $\left(K > -\dfrac{1}{2}\right)$.
*Note that the inequalities (13) and (14) are sharp.*

## 4. Subordination results for the class $M$

**Definition 1.** *A sequence* $\{b_k\}_{k=1}^{\infty}$ *of complex numbers is called a subordinating factor sequence, if whenever* $f(z)$ *is analytic, univalent and convex in* $U$, *we have the subordination given by*

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \tag{15}$$

*where* $z \in U$ *and* $a_1 = 1$.

**Lemma 1.** [17] *The sequence* $\{b_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \qquad (z \in U). \tag{16}$$

For brevity, let us denote

$$\sigma_k(\lambda, \mu, m, A, B) = [1 + (k-1)(\lambda - \mu + k\mu\lambda)]^m \{(k-1)(\lambda - \mu + k\mu\lambda) + |(A-B)b - B(k-1)(\lambda - \mu + k\mu\lambda)|\} B_k.$$

Let $\overline{M}$ be the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (3). Note that $\overline{M} \subseteq M$.

**Theorem 3.** *Let the function* $f(z)$ *defined by (1), be in the class* $\overline{M}$*, where* $-1 \leq A < B \leq 1$*. Also let* $\zeta$ *denote the familiar class of functions* $f(z) \in \mathcal{A}$ *which are also univalent and convex in* $U$*. Then*

$$\frac{\sigma_2(\lambda, \mu, m, A, B)}{2[(A-B)|b| + \sigma_2(\lambda, \mu, m, A, B)]}(f * g)(z) \prec g(z) \qquad (z \in U, g \in \zeta) \tag{17}$$

*and*

$$\Re(f(z)) > -\frac{(A-B)|b| + \sigma_2(\lambda, \mu, m, A, B)}{\sigma_2(\lambda, \mu, m, A, B)} \qquad (z \in U). \tag{18}$$

*In fact, the constant* $\frac{\sigma_2(\lambda,\mu,m,A,B)}{2[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}$ *is the best estimate.*

**Proof.** Let $f(z) \in \overline{M}$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \zeta$. Then

$$\frac{\sigma_2(\lambda, \mu, m, A, B)}{2[(A-B)|b| + \sigma_2(\lambda, \mu, m, A, B)]}(f * g)(z) = \frac{\sigma_2(\lambda, \mu, m, A, B)(z + \sum_{k=2}^{\infty} a_k b_k z^k)}{2[(A-B)|b| + \sigma_2(\lambda, \mu, m, A, B)]}.$$

Thus by the definition (15), the assertion of the theorem will hold if the sequence $\left\{\frac{\sigma_2(\lambda,\mu,m,A,B)a_k}{2[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}\right\}_{k=1}^{\infty}$ is a subordinating sequence with $a_1 = 1$. In view of Lemma 1 this will be true if and only if

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} \frac{\sigma_2(\lambda, \mu, m, A, B)}{2[(A-B)|b| + \sigma_2(\lambda, \mu, m, A, B)]} a_k z^k\right\} > 0 \qquad (z \in U). \tag{19}$$

Now

$$\Re\left\{1+\frac{\sigma_2(\lambda,\mu,m,A,B)}{[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}\sum_{k=1}^{\infty}a_kz^k\right\}=$$

$$\Re\left\{1+\frac{\sigma_2(\lambda,\mu,m,A,B)}{[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}a_1z+\frac{\sigma_2(\lambda,\mu,m,A,B)}{[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}\sum_{k=2}^{\infty}a_kz^k\right\}$$

$$\geq 1-\Re\left\{\left|\frac{\sigma_2(\lambda,\mu,m,A,B)}{[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}\right|r+\frac{\sum_{k=2}^{\infty}\sigma_k(\lambda,\mu,m,A,B)|a_k|r^k}{[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}\right\}.$$

Since $\sigma_k(\lambda,\mu,m,A,B)$ is a real increasing function of $k \qquad (k\geq 2)$,

$$1-\Re\left\{\left|\frac{\sigma_2(\lambda,\mu,m,A,B)}{[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}\right|r+\frac{\sum_{k=2}^{\infty}\sigma_k(\lambda,\mu,m,A,B)|a_k|r^k}{[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}\right\}$$

$$\geq 1-\left\{\frac{\sigma_2(\lambda,\mu,m,A,B)}{2[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}r+\frac{(A-B)|b|}{(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)}r\right\}$$

$=1-r>0.$

Thus (19) holds in $U$. This proves the inequality (17). The inequality (18) follows by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$ in (17). To prove the sharpness of the constant $\frac{\sigma_2(\lambda,\mu,m,A,B)}{2[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}$, we consider $f_0z\in\overline{M}$ given by

$$f_0(z)=z-\frac{(A-B)b}{\sigma_2(\lambda,\mu,m,A,B)}z^2.$$

Thus from (17) we have

$$\frac{\sigma_2(\lambda,\mu,m,A,B)}{2[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}f_0(z)\prec\frac{z}{1-z}.$$

It can be easily verified that

$$\min\left\{\Re\left(\frac{\sigma_2(\lambda,\mu,m,A,B)}{2[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}f_0(z)\right)\right\}=-\frac{1}{2}.$$

This shows that the constant $\frac{\sigma_2(\lambda,\mu,m,A,B)}{2[(A-B)|b|+\sigma_2(\lambda,\mu,m,A,B)]}$ is the best possible.

For the sake of completeness, we state some of the new and various other known results by specializing the parameters involved in Theorem 3.

**Corollary 5** *Let the function $f \in \mathcal{H}^m(b; A, B)$ satisfy the condition (4). Then*

$$\frac{2^{m-1}\{1+ \mid (A-B)b - B \mid \}}{(A-B) \mid b \mid +2^m\{1+ \mid (A-B)b - B \mid \}}(f * g)(z) \prec g(z) \tag{20}$$

$$(z \in \mathcal{U};\ m \in \mathbb{N}_0;\ g \in \mathcal{C})$$

*and*

$$Re\, f(z) > -\frac{(A-B) \mid b \mid +2^m\{1+ \mid (A-B)b - B \mid \}}{2^m\{1+ \mid (A-B)b - B \mid \}}, \quad (z \in \mathcal{U}).$$

*In addition, the constant factor*

$$\frac{2^{m-1}\{1+ \mid (A-B)b - B \mid \}}{(A-B) \mid b \mid +2^m\{1+ \mid (A-B)b - B \mid \}}$$

*in the subordination result (20) cannot be replaced by a larger one.*

**Corollary 6**[7] *Let the function $f \in \mathcal{A}$ belong to $\mathcal{S}_m^*(\alpha)$ satisfy the condition*

$$\sum_{n=2}^{\infty} \left(n^{m+1} - \alpha n^m\right)|a_n| \leq 1 - \alpha, \quad 0 \leq \alpha < 1.$$

*Then*

$$\frac{2^m - \alpha 2^{m-1}}{(1-\alpha) + (2^{m+1} - \alpha 2^m)}(f * g)(z) \prec g(z) \tag{21}$$

$$(z \in \mathcal{U};\ m \in \mathbb{N}_0;\ g \in \mathcal{C})$$

*and*

$$Re\, f(z) > -\frac{(1-\alpha) + (2^{m+1} - \alpha 2^m)}{2^{m+1} - \alpha 2^m} \quad (z \in \mathcal{U}).$$

*The constant factor*

$$\frac{2^m - \alpha 2^{m-1}}{(1-\alpha) + (2^{m+1} - \alpha 2^m)}$$

*in the subordination result (21) cannot be replaced by a larger one.*

**Corollary 7** [7] *Let the function $f \in \mathcal{A}$ belong to $\mathcal{C}(\alpha)$ satisfy the condition*

$$\sum_{n=2}^{\infty} n\left(n - \alpha\right)|a_n| \leq 1 - \alpha, \quad 0 \leq \alpha < 1.$$

*Then*

$$\frac{2-\alpha}{5-3\alpha}(f * g)(z) \prec g(z) \tag{22}$$

$$(z \in \mathcal{U};\ m \in \mathbb{N}_0;\ g \in \mathcal{C})$$

*and*

$$Re\, f(z) > -\frac{5-3\alpha}{2(2-\alpha)} \quad (z \in \mathcal{U}).$$

*The constant factor* $\frac{2-\alpha}{5-3\alpha}$ *in the subordination result (22) cannot be replaced by a larger one.*

**Corollary 8** [7] it Let the function $f \in \mathcal{A}$ belong to $\mathcal{S}^*(\alpha)$ satisfy the condition

$$\sum_{n=2}^{\infty} \big(n-\alpha\big)|a_k| \leq 1-\alpha, \quad 0 \leq \alpha < 1.$$

Then

$$\frac{2-\alpha}{2(3-2\alpha)}(f * g)(z) \prec g(z) \tag{23}$$

$$(z \in \mathcal{U};\ m \in \mathbb{N}_0;\ g \in \mathcal{C})$$

and $Re\, f(z) > -\frac{3-2\alpha}{(2-\alpha)} \quad (z \in \mathcal{U})$. The constant factor $\frac{2-\alpha}{2(3-2\alpha)}$ in the subordination result (23) cannot be replaced by a larger one.

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