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CERTAIN STRONG DIFFERENTIAL SUPERORDINATIONS USING SĂLĂGEAN AND RUSCHEWEYH OPERATORS

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Abstract. In the present paper we establish several strong differential superordinations regardind the new operator SR^m defined by convolution product of the extended Sălăgean operator and Ruscheweyh derivative, $SR^m : \mathcal{A}^*_\zeta \to \mathcal{A}^*_\zeta$, $SR^m f(z,\zeta) = (S^m * R^m) f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $R^m f(z,\zeta)$ denote the extended Ruscheweyh derivative, $S^m f(z,\zeta)$ is the extended Sălăgean operator and $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$, with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_\zeta$, is the class of normalized analytic functions.

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1. Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$

with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_\zeta$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Denote by

$$K_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}) : \operatorname{Re} \frac{z f''_z(z,\zeta)}{f'_z(z,\zeta)} + 1 > 0\}$$

the class of convex function in $U \times \overline{U}$.

We also extend the known differential operators to the new class of analytic functions $\mathcal{A}^*_\zeta$ introduced in [5].

Definition No. 1 *[1] For $f \in \mathcal{A}^*_\zeta$, $m \in \mathbb{N}$, the extended operator S^m is defined by $S^m : \mathcal{A}^*_\zeta \to \mathcal{A}^*_\zeta$,*

$$\begin{aligned} S^0 f(z,\zeta) &= f(z,\zeta), \\ S^1 f(z,\zeta) &= z f'_z(z,\zeta), ..., \\ S^{m+1} f(z,\zeta) &= z\left(S^m f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark No. 1 *[1] If $f \in \mathcal{A}^*_\zeta$, $f(z,\zeta) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then $S^m f(z,\zeta) = z + \sum_{j=2}^\infty j^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Definition No. 2 *[1] For $f \in \mathcal{A}^*_\zeta$, $m \in \mathbb{N}$, the extended operator R^m is defined by $R^m : \mathcal{A}^*_\zeta \to \mathcal{A}^*_\zeta$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta), \\ R^1 f(z,\zeta) &= z f'_z(z,\zeta), ..., \\ (m+1) R^{m+1} f(z,\zeta) &= z\left(R^m f(z,\zeta)\right)'_z + m R^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark No. 2 *[1] If $f \in \mathcal{A}^*_\zeta$, $f(z,\zeta) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then $R^m f(z,\zeta) = z + \sum_{j=2}^\infty C^m_{m+j-1} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [4].

Definition No. 3 *[4] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark No. 3 *[4] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 3 is equivalent to $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$, and $H\left(U \times \overline{U}\right) \subset f\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition No. 4 *[3] We denote by Q^* the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where $E(f,\zeta) = \{y \in \partial U : \lim_{z \to y} f(z,\zeta) = \infty\}$, and are such that $f'_z(y,\zeta) \neq 0$ for $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta) = a$ is denoted by $Q^*(a)$.*

We have need the following lemmas to study the strong differential superordinations.

Lemma No. 1 *[3] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\,\gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$*

is univalent in $U \times \overline{U}$ *and*

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. *The function* q *is convex and is the best subordinant.*

Lemma No. 2 *[3] Let* $q(z,\zeta)$ *be a convex function in* $U \times \overline{U}$ *and let* $h(z,\zeta) = q(z,\zeta) + \frac{1}{\gamma} z q'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, *where* $Re\ \gamma \geq 0$.

If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$ *is univalent in* $U \times \overline{U}$ *and*

$$q(z,\zeta) + \frac{1}{\gamma} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. *The function* q *is the best subordinant.*

2. Main results

Definition No. 5 *[2] Let* $m \in \mathbb{N} \cup \{0\}$. *Denote by* SR^m *the operator given by the Hadamard product (the convolution product) of the extended Sălăgean operator* S^m *and the extended Ruscheweyh operator* R^m, $SR^m : \mathcal{A}^*_\zeta \to \mathcal{A}^*_\zeta$,

$$SR^m f(z,\zeta) = (S^m * R^m) f(z,\zeta).$$

Remark No. 4 *[2] If* $f \in \mathcal{A}^*_\zeta$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then* $SR^m f(z,\zeta) = z + \sum_{j=2}^{\infty} C^m_{m+j-1} j^m a_j^2(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Theorem No. 1 *Let* $h(z,\zeta)$ *be a convex function in* $U \times \overline{U}$ *with* $h(0,\zeta) = 1$. *Let* $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta) dt$, $z \in U$, $\zeta \in \overline{U}$, $Re c > -2$, *and suppose that* $(SR^m f(z,\zeta))'_z$ *is univalent in* $U \times \overline{U}$, $(SR^m F(z,\zeta))'_z \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$ *and*

$$h(z,\zeta) \prec\prec (SR^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{1}$$

then

$$q(z,\zeta) \prec\prec (SR^m F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}} \int_0^z h(t,\zeta) t^{c+1} dt$. *The function* q *is convex and it is the best subordinant.*

Proof. We have

$$z^{c+1} F(z,\zeta) = (c+2) \int_0^z t^c f(t,\zeta)\, dt$$

and differentiating it, with respect to z, we obtain $(c+1) F(z,\zeta) + z F'_z(z,\zeta) = (c+2) f(z,\zeta)$ and

$$(c+1) SR^m F(z,\zeta) + z \left(SR^m F(z,\zeta)\right)'_z = (c+2) SR^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Differentiating the last relation with respect to z we have

$$\left(SR^m F(z,\zeta)\right)'_z + \frac{1}{c+2} z \left(SR^m F(z,\zeta)\right)''_{z^2} = \left(SR^m f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{2}$$

Using (2), the strong differential superordination (1) becomes

$$h(z,\zeta) \prec\prec \left(SR^m F(z,\zeta)\right)'_z + \frac{1}{c+2} z \left(SR^m F(z,\zeta)\right)''_{z^2}. \tag{3}$$

Denote

$$p(z,\zeta) = \left(SR^m F(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{4}$$

Replacing (4) in (3) we obtain

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1 for $n = 1$ and $\gamma = c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec \left(SR^m F(z,\zeta)\right)'_z,\ z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}} \int_0^z h(t,\zeta) t^{c+1} dt$. The function q is convex and it is the best subordinant.

Corollary No. 1 *Let* $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$, *where* $\beta \in [0,1)$. *Let* $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta)\, dt$, $z \in U$, $\zeta \in \overline{U}$, $Rec > -2$, *and suppose that* $\left(SR^m f(z,\zeta)\right)'_z$ *is univalent in* $U \times \overline{U}$, $\left(SR^m F(z,\zeta)\right)'_z \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$ *and*

$$h(z,\zeta) \prec\prec \left(SR^m f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{5}$$

then

$$q(z,\zeta) \prec\prec \left(SR^m F(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{(c+2)(1+\zeta-2\beta)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{t+1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 1 and considering $p(z,\zeta) = (SR^m F(z,\zeta))'_z$, the strong differential superordination (5) becomes

$$h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + \frac{1}{c+2} z p'_z(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

By using Lemma 1 for $n = 1$ and $\gamma = c+2$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{c+2}{z^{c+2}} \int_0^z h(t,\zeta) t^{c+1} dt = \frac{c+2}{z^{c+2}} \int_0^z \frac{1+(2\beta-\zeta)t}{1+t} t^{c+1} dt$$

$$= 2\beta - \zeta + \frac{(c+2)(1+\zeta-2\beta)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{t+1} dt \prec\prec (SR^m F(z,\zeta))'_z, \quad z \in U, \ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem No. 2 *Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q'_z(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$, $Re c > -2$. Let $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta) dt$, $z \in U$, $\zeta \in \overline{U}$, and suppose that $(SR^m f(z,\zeta))'_z$ is univalent in $U \times \overline{U}$, $(SR^m F(z,\zeta))'_z \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$ and*

$$h(z,\zeta) \prec\prec (SR^m f(z,\zeta))'_z, \quad z \in U, \ \zeta \in \overline{U}, \tag{6}$$

then

$$q(z,\zeta) \prec\prec (SR^m F(z,\zeta))'_z, \quad z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}} \int_0^z h(t,\zeta) t^{c+1} dt$. The function q is the best subordinant.

Proof. We obtain that

$$z^{c+1} F(z,\zeta) = (c+2) \int_0^z t^c f(t,\zeta) dt. \tag{7}$$

Differentiating (7), with respect to z, we have $(c+1) F(z,\zeta) + z F'_z(z,\zeta) = (c+2) f(z,\zeta)$ and

$$(c+1) SR^m F(z,\zeta) + z (SR^m F(z,\zeta))'_z = (c+2) SR^m f(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}. \tag{8}$$

Differentiating (8) with respect to z we have

$$(SR^m F(z,\zeta))'_z + \frac{1}{c+2} z (SR^m F(z,\zeta))''_{z^2} = (SR^m f(z,\zeta))'_z, \quad z \in U, \ \zeta \in \overline{U}. \tag{9}$$

Using (9), the strong differential superordination (6) becomes

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q'_z(z,\zeta) \prec\prec (SR^m F(z,\zeta))'_z + \frac{1}{c+2} z (SR^m F(z,\zeta))''_{z^2}. \tag{10}$$

Denote

$$p(z,\zeta) = (SR^m F(z,\zeta))'_z, \quad z \in U, \ \zeta \in \overline{U}. \tag{11}$$

Replacing (11) in (10) we obtain

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2} z p'_z(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

Using Lemma 2 for $n = 1$ and $\gamma = c + 2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \ z \in U, \ \zeta \in \overline{U}, \text{ i.e. } q(z,\zeta) \prec\prec (SR^m F(z,\zeta))'_z, \ z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}} \int_0^z h(t,\zeta) t^{c+1} dt$. The function q is the best subordinant.

Theorem No. 3 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$ and suppose that $(SR^m f(z,\zeta))'_z$ is univalent and $\frac{SR^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec (SR^m f(z,\zeta))'_z, \quad z \in U, \ \zeta \in \overline{U}, \tag{12}$$

then

$$q(z,\zeta) \prec\prec \frac{SR^m f(z,\zeta)}{z}, \quad z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z} \int_0^z h(t,\zeta) dt$. The function q is convex and it is the best subordinant.

Proof. Consider
$p(z,\zeta) = \frac{SR^m f(z,\zeta)}{z} = \frac{z + \sum_{j=2}^{\infty} C^m_{m+j-1} j^m a_j^2(\zeta) z^j}{z} = 1 + \sum_{j=2}^{\infty} C^m_{m+j-1} j^m a_j^2(\zeta) z^{j-1}$.
Evidently $p \in \mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta) + z p'_z(z,\zeta) = (SR^m f(z,\zeta))'_z$.

Then (12) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + z p'_z(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

By using Lemma 1 for $n = 1$ and $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \quad \text{i.e.} \quad q(z,\zeta) \prec\prec \frac{SR^m f(z,\zeta)}{z}, \quad z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Corollary No. 2 *Let $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\leq\beta<1$. Let $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_\zeta$ and suppose that $(SR^m f(z,\zeta))'_z$ is univalent and $\frac{SR^m f(z,\zeta)}{z}\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec(SR^m f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{13}$$

then

$$q(z,\zeta)\prec\prec\frac{SR^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{1+\zeta-2\beta}{z}\ln(1+z)$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 3 and considering $p(z,\zeta)=\frac{SR^m f(z,\zeta)}{z}$, the strong differential superordination (13) becomes

$$h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp'_z(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1 for $n=1$ and $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt=\frac{1}{z}\int_0^z\frac{1+(2\beta-\zeta)\,t}{1+t}dt$$

$$=2\beta-\zeta+\frac{1+\zeta-2\beta}{z}\ln(1+z)\prec\prec\frac{SR^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem No. 4 *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$. If $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_\zeta$, suppose that $(SR^m f(z,\zeta))'_z$ is univalent, $\frac{SR^m f(z,\zeta)}{z}\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec(SR^m f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{14}$$

then

$$q(z,\zeta)\prec\prec\frac{SR^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is the best subordinant.

Proof. Let
$p(z,\zeta) = \frac{SR^m f(z,\zeta)}{z} = \frac{z+\sum_{j=2}^{\infty} C_{m+j-1}^m j^m a_j^2(\zeta) z^j}{z} = 1 + \sum_{j=2}^{\infty} C_{m+j-1}^m j^m a_j^2(\zeta) z^{j-1}$.
Evidently $p \in \mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta) + zp_z'(z,\zeta) = (SR^m f(z,\zeta))_z'$, $z \in U$, $\zeta \in \overline{U}$, and (14) becomes

$$q(z,\zeta) + zq_z'(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 2 for $n = 1$ and $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.}$$
$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \frac{SR^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant.

Theorem No. 5 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}_\zeta^*$ and suppose that $\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)} \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{15}$$

then

$$q(z,\zeta) \prec\prec \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Proof. Consider
$p(z,\zeta) = \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)} = \frac{z+\sum_{j=2}^{\infty} C_{m+j}^{m+1} j^{m+1} a_j^2(\zeta) z^j}{z+\sum_{j=2}^{\infty} C_{m+j-1}^{m} j^{m} a_j^2(\zeta) z^j} = \frac{1+\sum_{j=2}^{\infty} C_{m+j}^{m+1} j^{m+1} a_j^2(\zeta) z^{j-1}}{1+\sum_{j=2}^{\infty} C_{m+j-1}^{m} j^{m} a_j^2(\zeta) z^{j-1}}$.
Evidently $p \in \mathcal{H}^*[1,1,\zeta]$.

We have $p_z'(z,\zeta) = \frac{\left(SR^{m+1}f(z,\zeta)\right)_z'}{SR^m f(z,\zeta)} - p(z,\zeta) \cdot \frac{(SR^m f(z,\zeta))_z'}{SR^m f(z,\zeta)}$ and $p(z,\zeta) + zp_z'(z,\zeta) = \left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)_z'$.

Then (15) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $n = 1$ and $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)},\ z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Corollary No. 3 *Let $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \le \beta < 1$. Let $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$ and suppose that $\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z$ is univalent, $\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)} \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \tag{16}$$

then

$$q(z,\zeta) \prec\prec \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}, \qquad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{1+\zeta-2\beta}{z}\ln(1+z)$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 5 and considering $p(z,\zeta) = \frac{SR^m f(z,\zeta)}{z}$, the strong differential superordination (16) becomes

$$h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $n = 1$ and $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)\,dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-\zeta)t}{1+t}dt$$

$$= 2\beta - \zeta + \frac{1+\zeta-2\beta}{z}\ln(1+z) \prec\prec \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}, \qquad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem No. 6 *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by $h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta)$. If $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$, suppose that $\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z$ is* *univalent,* $\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z)} \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$ *and satisfies the strong differential superordination*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec \left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \tag{17}$$

then

$$q(z,\zeta) \prec\prec \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. *The function* q *is the best subordinant.*

Proof. Let
$p(z,\zeta) = \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)} = \frac{z+\sum_{j=2}^{\infty} C_{m+j}^{m+1} j^{m+1} a_j^2(\zeta) z^j}{z+\sum_{j=2}^{\infty} C_{m+j-1}^{m} j^m a_j^2(\zeta) z^j} = \frac{1+\sum_{j=2}^{\infty} C_{m+j}^{m+1} j^{m+1} a_j^2(\zeta) z^{j-1}}{1+\sum_{j=2}^{\infty} C_{m+j-1}^{m} j^m a_j^2(\zeta) z^{j-1}}$.
Evidently $p \in \mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta)+zp'_z(z,\zeta) = \left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z$, $z \in U$, $\zeta \in \overline{U}$, and (17) becomes

$$q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 2 for $n = 1$ and $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.}$$
$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}, \ z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant.

Theorem No. 7 *Let* $h(z,\zeta)$ *be a convex function,* $h(0,\zeta) = 1$. *Let* $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}_\zeta^*$ *and suppose that* $\frac{1}{z}SR^{m+1}f(z,\zeta)$ *is univalent and* $(SR^m f(z,\zeta))'_z \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$. *If*

$$h(z,\zeta) \prec\prec \frac{1}{z}SR^{m+1}f(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}, \tag{18}$$

then

$$q(z,\zeta) \prec\prec (SR^m f(z,\zeta))'_z, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{m+1}{z^{m+1}}\int_0^z h(t,\zeta)t^m dt$. *The function* q *is convex and it is the best subordinant.*

Proof. With notation $p(z,\zeta) = (SR^m f(z,\zeta))'_z = 1+\sum_{j=2}^{\infty} C_{m+j-1}^{m} j^{m+1} a_j^2(\zeta) z^{j-1}$ and $p(0,\zeta) = 1$, we obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, $p(z,\zeta) + \frac{1}{m+1}zp'_z(z,\zeta) = \frac{1}{z}SR^{m+1}f(z,\zeta)$. Evidently $p \in \mathcal{H}^*[1,1,\zeta]$.

Then (18) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $n = 1$ and $\gamma = m+1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec \left(SR^m f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{m+1}{z^{m+1}} \int_0^z h(t,\zeta) t^m dt$. The function q is convex and it is the best subordinant.

Corollary No. 4 *Let $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \le \beta < 1$. Let $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$ and suppose that $\frac{1}{z} SR^{m+1} f(z,\zeta)$ is univalent and $\left(SR^m f(z,\zeta)\right)'_z \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \frac{1}{z} SR^{m+1} f(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}, \tag{19}$$

then

$$q(z,\zeta) \prec\prec \left(SR^m f(z,\zeta)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{(1+\zeta-2\beta)(m+1)}{z^{m+1}} \int_0^z \frac{t^m}{1+t} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 7 and considering $p(z,\zeta) = \left(SR^m f(z,\zeta)\right)'_z$, the strong differential superordination (19) becomes

$$h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $n = 1$ and $\gamma = m+1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{m+1}{z^{m+1}} \int_0^z h(t,\zeta) t^m dt = \frac{m+1}{z^{m+1}} \int_0^z t^m \frac{1+(2\beta-\zeta)t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{(1+\zeta-2\beta)(m+1)}{z^{m+1}} \int_0^z \frac{t^m}{1+t} dt \prec\prec \left(SR^m f(z,\zeta)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem No. 8 *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by $h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} z q'_z(z,\zeta)$. If $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$, suppose that $\frac{1}{z} SR^{m+1} f(z,\zeta)$*

is univalent, $(SR^m f(z,\zeta))'_z \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$ *and satisfies the strong differential superordination*

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} z q'_z(z,\zeta) \prec\prec \frac{1}{z} SR^{m+1} f(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \quad (20)$$

then

$$q(z,\zeta) \prec\prec (SR^m f(z,\zeta))', \quad z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{m+1}{z^{m+1}} \int_0^z h(t,\zeta) t^m dt$. *The function* q *is the best subordinant.*

Proof. Let $p(z,\zeta) = (SR^m f(z,\zeta))'_z = 1 + \sum_{j=2}^{\infty} C_{m+j-1}^m j^{m+1} a_j^2(\zeta) z^{j-1}$.

Differentiating with respect to z, we obtain $p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta) = \frac{1}{z} SR^{m+1} f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and (20) becomes

$$q(z,\zeta) + \frac{1}{m+1} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

Using Lemma 2 for $n = 1$ and $\gamma = m+1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \quad \text{i.e.}$$

$$q(z,\zeta) = \frac{m+1}{z^{m+1}} \int_0^z h(t,\zeta) t^m dt \prec\prec (SR^m f(z,\zeta))', \quad z \in U, \ \zeta \in \overline{U},$$

and q is the best subordinant.

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