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SUBCLASSES OF P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

M.K.Aouf

Abstract. In this paper we introduce two subclasses $T^*(A,B,\alpha,p,j)$ and $C(A,B,\alpha,p,j)$ of analytic and p- valent functions with negative coefficients. We obtain coefficient estimates, distortion theorems , extreme points and radii of close - to - convexity, starlikeness and convexity of order $\phi (0 \leq \phi < p)$ for these classes. We also obtain integral operators for these classes. Furthermore, several results for the modified Hadamard products of functions belonging to the classes $T^*(A,B,\alpha,p,j)$ and $C(A,B,\alpha,p,j)$ are also given.

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1. Introduction

Let $S(p)$ denote the class of functions of the form :

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1,2,...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$. Let the functions $f(z)$ and $g(z)$ be analytic in U . Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in U , with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = g(w(z)) (z \in U)$.We denote this subordination by $f(z) \prec g(z)$.

For A, B fixed, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $0 \leq \alpha < p-j+1$, $1 \leq j \leq p$ and $p \in N$, we say that $f(z) \in A(p)$ is in the class $S^*(A,B,\alpha,p,j)$ if it satisfies the following subordination condition:

$$\frac{zf^{(j)}(z)}{f^{(j-1)}(z)} \prec \frac{p-j+1+[(p-j+1)B+(A-B)(p-j+1-\alpha)]z}{1+Bz} \quad (z \in U), \tag{1.2}$$

or, equivalently, $f(z) \in S^*(A,B,\alpha,p,j)$ if and only if

$$\left| \frac{\dfrac{zf^{(j)}(z)}{f^{(j-1)}(z)} - (p-j+1)}{B\dfrac{zf^{(j)}(z)}{f^{(j-1)}(z)} - [(p-j+1)B + (A-B)(p-j+1-\alpha)]} \right| < 1 \;\; (z \in U). \tag{1.3}$$

Further $f(z) \in A(p)$ is said to belong to the class $K(A,B,\alpha,p,j)$ if and only if $\dfrac{zf^{(j)}(z)}{p-j+1} \in S^*(A,B,\alpha,p,j)$.

We note that :

(i) $S^*(A,B,0,p,j) = H^0_{p,j}(A,B)$, $K(A,B,0,p,j) = H^1_{p,j}(A,B)(-1 \le B < A \le 1\;;\; 1 \le j \le p)$, $S^*(-1,1,\alpha,p,j) = H^0_{p,j}(\alpha)$ *and* $K(-1,1,\alpha,p,j) = H^1_{p,j}(\alpha)(0 \le \alpha < p-j+1;\; 1 \le j \le p)$(Srivastava et al. [10]) (see also Nunokawa [6]);

(ii) $S^*(A,B,\alpha,p,j) = H^0_{p,j}(A,B,\alpha)$ and $K(A,B,\alpha,p,j) = H^1_{p,j}(A,B,\alpha)(-1 \le B < A \le 1\;;\; 1 \le j \le p)$ (Aouf [2]).

Let $T(p)$ denote the subclass of $S(p)$ consisting of functions of the form :

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \ge 0;\, p \in N). \tag{1.4}$$

Further , we define the classes $T^*(A,B,\alpha,p,j)$ and $C(A,B,\alpha,p,j)$ by

$$T^*(A,B,\alpha,p,j) = S^*(A,B,\alpha,p,j) \cap T(p) \tag{1.5}$$

and

$$C(A,B,\alpha,p,j) = K(A,B,\alpha,p,j) \cap T(p). \tag{1.6}$$

We note that, by specializing the parameters A,B,α,p and j, we obtain the following subclasses studied by various authors :

(i) $T^*(A,B,\alpha,p,1) = T^*_p(A,B,\alpha)$ and $C(A,B,0,p,1) = C_p(A,B,\alpha)(0 \le \alpha < p;\, p \in N)$ (Aouf [1]);

(ii) $T^*(A,B,0,p,1) = T^*_p(A,B)$ and $C(A,B,0,p,1) = C_p(A,B)$(Goel and Sohi [3]);

(iii)$T^*(-1,1,\alpha,p,1) = T^*(p,\alpha)$ and $C(-1,1,\alpha,p,1) = C(p,\alpha)(0 \le \alpha < p;\, p \in N)$(Owa [7]);

(iv) $T^*(-\beta,\beta,\alpha,1,1) = T^*(\alpha,\beta)$ and $C(-\beta,\beta,\alpha,1,1) = C(\alpha,\beta)(0 \le \alpha < 1\,;\, 0 < \beta \le 1)$(Gupta and Jain [5]);

(v) $T^*(-1,1,\alpha,1,1)=T^*(\alpha)$ and $C(-1,1,\alpha,1,1)=C(\alpha)$ $(0\le\alpha<1)$(Silverman [9]).

Also we note that :
$T^*(-A,-B,\alpha,p,p)=F_p^*(A,B,\alpha)$

$$=\left\{f(z)\in T(p):\left|\frac{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-1}{B\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-[B+(A-B)(1-\alpha)]}\right|<1\right.$$

$$\left.(z\in U,\ \ -1\le B<A\le 1\ \ ,-1\le B<0\ ,\ 0\le\alpha<1)\right\}. \tag{1.7}$$

In [4] Guney and Eker studied the class $A_0^*(p,A,B,\alpha)$, where $A_0^*(p,A,B,\alpha)$ is defined as follows:

$$A_0^*(p,A,B,\alpha)=\{f(z)\in T(p):$$

$$\left|\frac{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-1}{B\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-[B+(A-B)(p-\alpha)]}\right|<1$$

$$(z\in U,\ \ -1\le B<A\le 1,\ \ -1\le B<0,0\le\alpha<p)\}\,. \tag{1.8}$$

We note that this definition is not correct because $\left.\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right|_{z=0}=1$.Then we have

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\prec\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}$$

$$(z\in U,-1\le B<A\le 1,\ -1\le B<0\quad\text{and}\quad 0\le\alpha<1)$$

and

$$p-1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\prec p-1+\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}$$

$$=\frac{p+[pB+(A-B)(1-\alpha)]z}{1+Bz}.$$

Then $f(z)\in A_0^*(p,A,B,\alpha)$ if and only if (1.7) is satisfied .

2. Coefficient Estimates

Theorem 1 *.Let the function $f(z)$ be defined by* (1.4)*. Then $f(z) \in T^*(A, B, \alpha, p, j)$ if and only if*

$$\sum_{n=1}^{\infty}[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)a_{p+n}$$

$$\leq (B-A)(p-j+1-\alpha)\delta(p,j-1), \tag{2.1}$$

where

$$\delta(p,j)=\frac{p!}{(p-j)!}=\begin{cases} p(p-1).....(p-j+1) & (j\neq 0) \\ 1 & (j=0). \end{cases} \tag{2.2}$$

Proof. . Assume that the inequality (2.1) holds true and let $|z|=1$.Then we have

$$\left|zf^{(j)}(z)\ -\ (p-j+1)f^{(j-1)}(z)\right|-\Big|Bzf^{(j)}(z)-[(p-j+1)B+$$

$$(A-B)(p-j+1-\alpha)]f^{(j-1)}(z)\Big|$$

$$=\left|-\sum_{n=1}^{\infty}n\,\delta\,(p+n,j-1)a_{p+n}z^{p+n-j+1}\right|$$

$$-\Big|(B-A)(p-j+1-\alpha)\delta(p,j-1)z^{p-j+1}+$$

$$\sum_{n=1}^{\infty}[nB+\ (B-A)(p-j+1-\alpha)]\ \delta(p+n,j-1)a_{p+n}z^{p+n-j+1}\Big|$$

$$\leq\sum_{n=1}^{\infty}[n\,(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)a_{p+n}$$

$$-(B-A)(p-j+1-\alpha)\delta(p,j-1)\leq 0,$$

by hypothesis . Hence, by the maximum modulus theorem , we have $f(z) \in T^*(A, B, \alpha, p, j)$.

Conversely, let $f(z) \in T^*(A, B, \alpha, p, j)$ be given by (1.4). Then from (1.3) and (1.4), we have

$$\left| \frac{\dfrac{zf^{(j)}(z)}{f^{(j-1)}(z)} - (p-j+1)}{B\dfrac{zf^{(j)}(z)}{f^{(j-1)}(z)} - [(p-j+1)B + (A-B)(p-j+1-\alpha)]} \right|$$

$$= \left| \frac{-\sum_{n=1}^{\infty} n\delta(p+n,j-1)a_{p+n}z^n}{(B-A)(p-j+1-\alpha)\delta(p,j-1) - \sum_{n=1}^{\infty} [nB+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)a_{p+n}z^n} \right| < 1 \ (z \in U).$$

Since $|\text{Re}(z)| \le |z|$ for all z, we have

$$\text{Re}\left\{ \frac{\sum_{n=1}^{\infty} n\delta(p+n,j-1)a_{p+n}z^n}{(B-A)(p-j+1-\alpha)\delta(p,j-1) - \sum_{n=1}^{\infty} [nB+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)a_{p+n}z^n} \right\} < 1. \quad (2.3)$$

Choosing values of z on the real axis so that $\dfrac{zf^{(j)}(z)}{f^{(j-1)}(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=1}^{\infty} [n(1+B) + (B-A)(p-j+1-\alpha)]\delta(p+n,j-1)a_{p+n}$$

$$\le (B-A)(p-j+1-\alpha)\delta(p,j-1), \quad (2.4)$$

which leads us at once to (2.1). This completes the proof of Theorem 1.

Corollary 2 *.Let the function $f(z)$ defined by (1.4) be in the class $T^*(A,B,\alpha,p,j)$.*

Then we have

$$a_{p+n} \le \frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{[n(1+B) + (B-A)(p-j+1-\alpha)]\delta(p+n,j-1)} \ (p,n \in N). \quad (2.5)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{[n(1+B) + (B-A)(p-j+1-\alpha)]\delta(p+n,j-1)} z^{p+n} \ (p,n \in N). \quad (2.6)$$

Theorem 3 *.Let the function $f(z)$ defined by* (1.4)*. Then $f(z) \in C(A, B, \alpha, p, j)$ if and only if*

$$\sum_{n=1}^{\infty}[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j)a_{p+n}$$
$$\leq (B-A)(p-j+1-\alpha)\delta(p,j). \tag{2.7}$$

Proof. Since $f(z) \in C(A, B, \alpha, p, j)$ if and only if $\frac{zf^{(j)}(z)}{p-j+1} \in T^*(A, B, \alpha, p, j)$, we have Theorem 2 by replacing a_{p+n} by $(\frac{n+p-j+1}{p-j+1})a_{p+n}$ in Theorem 1.

Corollary 4 *.Let the function $f(z)$ defined by* (1.4) *be in the class $C(A, B, \alpha, p, j)$.*

Then we have

$$a_{p+n} \leq \frac{(B-A)(p-j+1-\alpha)\delta(p,j)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j)} \quad (p, n \in N). \tag{2.8}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)(p-j+1-\alpha)\delta(p,j)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j)} z^{p+n} \quad (p, n \in N). \tag{2.9}$$

3.Extreme Points

From Theorem 1 and Theorem 2, we see that both $T^*(A, B, \alpha, p, j)$ and $C(A, B, \alpha, p, j)$ are closed under convex linear combinations, which enables us to determine the extreme points for these classes.

Theorem 5 *.Let*

$$f_p(z) = z^p \tag{3.1}$$

and

$$f_{p+n}(z) = z^p - \frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)} z^{p+n}\, (p,n \in N). \tag{3.2}$$

Then $f(z) \in T^*(A,B,\alpha,p,j)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z), \tag{3.3}$$

where $\lambda_{p+n} \geq 0$ and $\sum\limits_{n=0}^{\infty} \lambda_{p+n} = 1$.

Proof. Suppose that

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z)$$

$$= z^p - \sum_{n=1}^{\infty} \frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)} \lambda_{p+n} z^{p+n}. \tag{3.4}$$

Then it follows that

$$\sum_{n=1}^{\infty} \frac{[n(1+B)+(B-A)(p-j+1-\alpha)\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)]\delta(p,j-1)} \cdot$$

$$\cdot \frac{(B-A)(p-j+1-\alpha)]\delta(p,j-1)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)} \lambda_{p+n}$$

$$= \sum_{n=1}^{\infty} \lambda_{p+n} = 1 - \lambda_p \leq 1. \tag{3.5}$$

Therefore, by Theorem 1, $f(z) \in T^*(A,B,\alpha,p,j)$.

Conversely, assume that the function $f(z)$ defined by (1.4) belongs to the class $T^*(A,B,\alpha,p,j)$. Then

$$a_{p+n} \leq \frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)} \quad (p,n \in N). \tag{3.6}$$

Setting

$$\lambda_{p+n} = \frac{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)} a_{p+n} \ \ (p,n \in N) \quad (3.7)$$

and

$$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{p+n}, \quad (3.8)$$

we see that $f(z)$ can be expressed in the form (3.3). This completes the proof of Theorem 3.

Corollary 6 *.The extreme points of the class $T^*(A,B,\alpha,p,j)$ are the fnctions $f_p(z) = z^p$ and*

$$f_{p+n}(z) = z^p - \frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)} z^{p+n} \ \ (p,n \in N).$$

Similarly, we have

Theorem 7 *.Let*

$$f_p(z) \ = z^p \quad (3.9)$$

and

$$f_{p+n}(z) = z^p - \frac{(B-A)(p-j+1-\alpha)\delta(p,j)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j)} z^{p+n} \ \ (p,n \in N). \quad (3.10)$$

Then $f(z) \in C(A,B,\alpha,p,j)$ if and only if it can be expressed in the form

$$f(z) \ = \sum_{n=0}^{\infty} \lambda_{p+n} f_{p+n}(z), \quad (3.11)$$

where $\lambda_{p+n} \geq 0$ and $\sum\limits_{n=0}^{\infty} \lambda_{p+n} = 1.$

Corollary 8 *.The extreme points of the class $C(A,B,\alpha,p,j)$ are the functions $f_p(z) = z^p$ and*

$$f_{p+n}(z) = z^p - \frac{(B-A)(p-j+1-\alpha)\delta(p,j)}{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j)} z^{p+n} \quad (p,n \in N).$$

4.Distortion Theorems

Theorem 9 *.Let the function* $f(z)$ *defined by* (1.4) *be in the class* $T^*(A,B,\alpha,p,j)$.

Then, for $|z| = r < 1$,

$$r^p - \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} r^{p+1} \leq |f(z)| \leq$$
$$r^p + \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} r^{p+1}, \tag{4.1}$$

and

$$pr^{p-1} - \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)]} r^p \leq \left| f'(z) \right| \leq$$
$$pr^{p-1} + \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)]} r^p. \tag{4.2}$$

The equality in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} z^{p+1} \quad (z = \pm r). \tag{4.3}$$

Proof. Since $f(z) \in T^*(A,B,\alpha,p,j)$, in view of Theorem 1, we have

$$[1+B+(B-A)(p-j+1-\alpha)]\delta(p+1,j-1) \sum_{n=1}^{\infty} a_{p+n}$$
$$\leq \sum_{n=1}^{\infty} [n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)$$
$$\leq (B-A)(p-j+1-\alpha)\delta(p,j-1),$$

which evidently yields

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)}. \tag{4.4}$$

Consequently, for $|z| = r < 1,$ we obtain

$$|f(z)| \leq r^p + r^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$

$$\leq r^p + \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} r^{p+1}$$

and

$$|f(z)| \geq r^p - r^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$

$$\geq r^p - \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} r^{p+1},$$

which prove the assertion (4.1) of Theorem 5.

Also from Theorem 1, it follows that

$$\sum_{n=1}^{\infty} (p+n) a_{p+n} \leq \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)]} \ . \tag{4.5}$$

Consequently, for $|z| = r < 1,$ we have

$$\left|f^{'}(z)\right| \leq pr^{p-1} + \sum_{n=1}^{\infty} (p+n) a_{p+n} r^{p+n-1}$$

$$\leq pr^{p-1} + r^p \sum_{n=1}^{\infty} (p+n) a_{p+n}$$

$$\leq pr^{p-1} + \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)]} r^p$$

and

$$\left|f^{'}(z)\right| \geq pr^{p-1} - \sum_{n=1}^{\infty} (p+n) a_{p+n}\, r^{p+n-1}$$

$$\geq pr^{p-1} - r^p \sum_{n=1}^{\infty} (p+n) a_{p+n}$$

$$\geq pr^{p-1} - \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)]} r^p,$$

which prove the assertion (4.2) of Theorem 5.

Finally, it is easy to see that the bounds in (4.1) and (4.2) are attained for the function $f(z)$ given already by (4.3).

Corollary 10 *.Let the function $f(z)$ defined by (1.4) be in the class $T^*(A,B,\alpha,p,j)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{(1+B)(p+1) + (B-A)(p-j+1-\alpha)(j-1)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)}. \tag{4.6}$$

The result is sharp, with the extremal function $f(z)$ given (4.3).

Theorem 11 *. Let the function $f(z)$ defined by (1.4) be in the class $C(A,B,\alpha,p,j)$.*

Then, for $|z| = r < 1$,

$$r^p - \frac{(B-A)(p-j+1-\alpha)(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} r^{p+1} \leq |f(z)| \leq$$

$$r^p + \frac{(B-A)(p-j+1-\alpha)(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} r^{p+1} \tag{4.7}$$

and

$$pr^{p-1} - \frac{(B-A)(p-j+1-\alpha)(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)]} r^p \leq \left| f'(z) \right| \leq$$

$$pr^{p-1} + \frac{(B-A)(p-j+1-\alpha)(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)]} r^p. \tag{4.8}$$

The results are sharp.

Proof. . The proof of Theorem 6 is abtained by using the same technique as in the proof of Theorem 5 with the aid of Theorem 2. Further we can show that the bounds of Theorem 6 are sharp for the function $f(z)$ defined by

$$f(z) = z^p - \frac{(B-A)(p-j+1-\alpha)(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} z^{p+1} \quad (p \in N). \tag{4.9}$$

Corollary 12 *.Let the function $f(z)$ defined by* (1.4) *be in the class* $C(A,B,\alpha,p,j)$. *Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{(1+B)(p+1)+(B-A)(p-j+1-\alpha)j}{[1+B+(B-A)(p-j+1-\alpha)](p+1)}. \tag{4.10}$$

The result is sharp, with the extremal function $f(z)$ given by (4.9).

4.Integral Operators

Theorem 13 *. Let the function $f(z)$ defined by* (1.4) *be in the class* $T^*(A,B,\alpha,p,j)$,

and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \tag{5.1}$$

also belongs to the class $T^*(A,B,\alpha,p,j)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \tag{5.2}$$

where

$$b_{p+n} = (\frac{c+p}{c+p+n})a_{p+n}.$$

Therefore

$$\sum_{n=1}^{\infty}[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)b_{p+n}$$

$$= \sum_{n=1}^{\infty}[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)(\frac{c+p}{c+p+n})a_{p+n}$$

$$\leq \sum_{n=1}^{\infty}[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)a_{p+n}$$

$$\leq (B-A)(p-j+1-\alpha)\delta(p,j-1),$$

scince $f(z) \in T^*(A,B,\alpha,p,j)$.Hence, by Theorem 1, $F(z) \in T^*(A,B,\alpha,p,j)$.

Corollary 14 *.Under the same conditions as Theorem 7, a similar proof shows that the function $F(z)$ defined by* (5.1) *is in the class $C(A,B,\alpha,p,j)$, wherever $f(z)$ is in the class $C(A,B,\alpha,p,j)$.*

6.Radii of Close - to- Convexity, Starlikeness and Convexity

Theorem 15 *.Let the function $f(z)$ defined by* (1.4) *be in the class $T^*(A,B,\alpha,p,j)$, then $f(z)$ is p-valently close - to - convex of order ϕ $(0 \leq \phi < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_n \left\{ \frac{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)} (\frac{p-\phi}{p+n}) \right\}^{\frac{1}{n}} \ (n \geq 1). \tag{6.1}$$

The result is sharp, with the extremal function $f(z)$ given (2.6).

Proof. We must show that $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \phi$ for $|z| < r_1$. We have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{n=1}^{\infty} (p+n) a_{p+n} |z|^n .$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \phi$ if

$$\sum_{n=1}^{\infty} (\frac{p+n}{p-\phi}) a_{p+n} |z|^n \leq 1. \tag{6.2}$$

Hence, by Theorem 1, (6.2) will be true if

$$(\frac{p+n}{p-\phi}) |z|^n \leq$$

$$\frac{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}$$

or if

$$|z| \leq \left\{ \frac{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)} (\frac{p-\phi}{p+n}) \right\}^{\frac{1}{n}} \ (n \geq 1). \tag{6.3}$$

The theorem follows easily from (6.3).

Theorem 16 . *Let the function $f(z)$ defined by* (1.4) *be in the class* $T^*(A,B,\alpha,p,j)$, *then $f(z)$ is p-valently starlike of order $\phi\,(0 \le \phi < p)$ in $|z| \le r_2$, where*

$$r_2 = \inf_n \left\{ \frac{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)} \left(\frac{p-\phi}{p+n-\varphi}\right) \right\}^{\frac{1}{n}} \ (n \ge 1). \tag{6.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.6).
Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \le p - \phi \qquad \text{for} \quad |z| < r_2.$$

Using similar arguments as given by Theorem 9, we can get the following result.

Corollary 17 .*Let the function $f(z)$ defined by* (1.4) *be in the class* $T^*(A,B,\alpha,p,j)$,

then $f(z)$ is p-valently convex of order ϕ $(0 \le \phi < p)$ in $|z| < r_3$, where

$$r_3 = \inf_n \left\{ \frac{[n(1+B)+(B-A)(p-j+1-\alpha)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)} \left(\frac{p(p-\phi)}{(p+n)(p+n-\phi)}\right) \right\}^{\frac{1}{n}}$$

$$(n \ge 1). \tag{6.5}$$

The result is sharp, with the ertremal function $f(z)$ given by (2.6).

7.Modified Hadamard Products

Let the functions $f_\nu(z)(\nu = 1,2)$ be defined by

$$f_\nu(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,\nu} z^{p+n} \quad (a_{p+n,\nu} \ge 0; \ \ \nu = 1,2). \tag{7.1}$$

Then the modified Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n}. \tag{7.2}$$

Throughout this section, we assume further that

$$X(n,A,B,\alpha,p,j)=[n(1+B)+(B-A)(p-j+1-\alpha)]. \tag{7.3}$$

Theorem 18 . *Let the functions* $f_\nu(z)(\nu=1,2)$ *defined by* (7.1) *be in the class* $T^*(A,B,\alpha,p,j)$. *Then* $(f_1*f_2)(z)\in T^*(A,B,\gamma,p,j)$, *where*

$$\gamma=(p-j+1)-$$

$$\frac{(1+B)(B-A)(p-j+1-\alpha)^2(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)]^2(p+1)-(B-A)^2(p-j+1-\alpha)^2(p-j+2)}. \tag{7.4}$$

The result is sharp .

Proof. Employing the technique used earlier by Schild and Silverman [7], we need to find the largest γ such that

$$\sum_{n=1}^{\infty}\frac{X(n,A,B,\gamma,p,j)\delta(p+n,j-1)}{(B-A)(p-j+1-\gamma)\delta(p,j-1)}a_{p+n,1}a_{p+n,2}\le 1. \tag{7.5}$$

Since

$$\sum_{n=1}^{\infty}\frac{X(n,A,B,\alpha,p,j)\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}a_{p+n,1}\le 1 \tag{7.6}$$

and

$$\sum_{n=1}^{\infty}\frac{X(n,A,B,\alpha,p,j)\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}a_{p+n,2}\le 1, \tag{7.7}$$

by the Cauchy - Schwarz inequality we have

$$\sum_{n=1}^{\infty}\frac{X(n,A,B,\alpha,p,j)\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}\sqrt{a_{p+n,1}a_{p+n,2}}\le 1. \tag{7.8}$$

Thus it is sufficient to show that

$$\frac{X(n,A,B,\gamma,p,j)}{(p-j+1-\gamma)}a_{p+n,1}a_{p+n,2}\le\frac{X(n,A,B,\alpha,p,j)}{(p-j+1-\alpha)}\sqrt{a_{p+n,1}a_{p+n,2}}\ \ (n\in N), \tag{7.9}$$

that is, that

$$\sqrt{a_{p+n,1a_{p+n,2}}} \leq \frac{X(n,A,B,\alpha,p,j)(p-j+1-\gamma)}{X(n,A,B,\gamma,p,j)\delta(p-j+1-\alpha)} \quad (n \in N). \tag{7.10}$$

Note that

$$\sqrt{a_{p+n,1a_{p+n,2}}} \leq \frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{X(n,A,B,\alpha,p,j)\delta(p+n,j-1)} \quad (n \in N). \tag{7.11}$$

Consequently, we need only to prove that

$$\frac{(B-A)(p-j+1-\alpha)\delta(p,j-1)}{X(n,A,B,\alpha,p,j)\delta(p+n,j-1)} \leq$$

$$\frac{X(n,A,B,\alpha,p,j)(p-j+1-\gamma)}{X(n,A,B,\gamma,p,j)(p-j+1-\alpha)} \quad (n \in N) \tag{7.12}$$

or, equivalently, that

$$\gamma \leq (p-j+1)-$$

$$\frac{n(1+B)(B-A)(p-j+1-\alpha)^2\delta(p,j-1)}{[X(n,A,B,\alpha,p,j)]^2\delta(p+n,j-1)-(B-A)^2(p-j+1-\alpha)^2\delta(p,j-1)} \quad (n \in N). \tag{7.13}$$

Since

$$D(n) = (p-j+1)-$$

$$\frac{n(1+B)(B-A)(p-j+1-\alpha)^2\delta(p,j-1)}{[X(n,A,B,\alpha,p,j)]^2\delta(p+n,j-1)-(B-A)^2(p-j+1-\alpha)^2\delta(p,j-1)} \tag{7.14}$$

is an inereasing function of $n (n \in N)$, letting $n = 1$ in (7.14), we obtain

$$\gamma \leq D(1) = (p-j+1)-$$

$$\frac{(1+B)(B-A)(p-j+1-\alpha)^2(p-j+2)}{[X(1,A,B,\alpha,p,j)]^2(p+1)-(B-A)^2(p-j+1-\alpha)^2(p-j+2)}, \tag{7.15}$$

which completes the proof of Theorem 10.

Finally, by taking the functions

$$f_\nu(z) = z^p - \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} z^{p+1} \quad (\nu = 1,2; p \in N) \quad (7.16)$$

we can see that the result is sharp.

Corollary 19 *.For $f_\nu(z)(\nu = 1,2)$ as in Theorem 10, we have*

$$h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1}a_{p+n,2}} z^{p+n} \quad (7.17)$$

belongs to the class $T^*(A, B, \alpha, p, j)$.
The result follows from the inequality(7.8). It is sharp for the same functions as in Theorem 10.

Corollary 20 *.Let the functions $f_\nu(z)(\nu = 1,2)$ defined by (7.1) be in the class $C(A, B, \alpha, p, j)$.Then $(f_1 * f_2)(z) \in C(A, B, \lambda, p, j)$, where*

$$\lambda = (p-j+1) -$$

$$\frac{(1+B)(B-A)(p-j+1-\alpha)^2(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)]^2(p+1) - (B-A)^2(p-j+1-\alpha)^2(p-j+1)}. \quad (7.18)$$

The result is sharp for the functions

$$f_\nu(z) = z^p - \frac{(B-A)(p-j+1-\alpha)(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} z^{p+1} \quad (\nu = 1,2; p \in N). \quad (7.19)$$

Using arguments similar to those in the proof of Theorem 10, we obtain the following result .

Theorem 21 *.Let the function $f_1(z)$ defined by (7.1) be in the class $T^*(A, B, \alpha, p, j)$ and the function $f_2(z)$ defined by (7.1) be in the class $T^*(A, B, \tau, p, j)$. Then $(f_1 * f_2)(z) \in T^*(A, B, \zeta, p, j)$, where*

$$\zeta = (p-j+1)-$$

$$\frac{(1+B)(B-A)(p-j+1-\alpha)(p-j+1-\tau)(p-j+2)}{X(1,A,B,\alpha,p,j)X(1,A,B,\tau,p,j)(p+1)-\Omega(p-j+2)}, \tag{7.20}$$

where

$$\Omega = (B-A)^2(p-j+1-\alpha)(p-j+1-\tau). \tag{7.21}$$

The result is the best possible for the functions

$$f_1(z) = z^p - \frac{(B-A)(p-j+1-\alpha)(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} z^{p+1} \quad (p \in N) \tag{7.22}$$

and

$$f_2(z) = z^p - \frac{(B-A)(p-j+1-\tau)(p-j+2)}{[1+B+(B-A)(p-j+1-\tau)](p+1)} z^{p+1} \quad (p \in N). \tag{7.23}$$

Corollary 22 . *Let the function $f_1(z)$ defined by (7.1) be in the class $C(A,B,\alpha,p,j)$*

and let the function $f_2(z)$ defined by (7.1) be in the class $C(A,B,\tau,p,j)$. Then $(f_1 * f_2)(z) \in C(A,B,\theta,p,j)$, where

$$\theta = (p-j+1)-$$

$$\frac{(1+B)(B-A)(p-j+1-\alpha)(p-j+1-\tau)(p-j+1)}{X(1,A,B,\alpha,p,j)X(1,A,B,\tau,p,j)(p+1)-\Omega(p-j+1)}, \tag{7.24}$$

where Ω is defined by (7.21). The result is sharp for the functions

$$f_1(z) = z^p - \frac{(B-A)(p-j+1-\alpha)(p-j+1)}{[1+B+(B-A)(p-j+1-\alpha)](p+1)} z^{p+1} \quad (p \in N) \tag{7.25}$$

and

$$f_2(z) = z^p - \frac{(B-A)(p-j+1-\tau)(p-j+1)}{[1+B+(B-A)(p-j+1-\tau)](p+1)} z^{p+1} \quad (p \in N). \tag{7.26}$$

Theorem 23 .*Let the functions* $f_\nu(z)(\nu = 1,2)$ *defined by* (7.1) *be in the class* $T^*(A,B,\alpha,p,j)$.*Then the function*

$$h(z) = z^p - \sum_{n=1}^{\infty}(a_{p+n,1}^2 + a_{p+n,2}^2)z^{p+n} \tag{7.27}$$

belongs to the class $T^*(A,B,\varphi,p,j)$, where

$$\varphi = (p-j+1)-$$

$$\frac{2(1+B)(B-A)(p-j+1-\alpha)^2(p-j+2)}{[1+B+(B-A)(p-j+1-\alpha)]^2(p+1)-2(B-A)^2(p-j+1-\alpha)^2(p-j+2)}. \tag{7.28}$$

The result is sharp for the functions $f_\nu(z)(\nu = 1,2)$ defined by (7.16).
Proof. By virtue of Theorem 1, we obtain

$$\sum_{n=1}^{\infty}\left\{\frac{[X(n,A,B,\alpha,p,j)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}\right\}^2 a_{p+n,\nu}^2$$

$$\leq \left\{\sum_{n=1}^{\infty}\frac{[X(n,A,B,\alpha,p,j)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}a_{p+n,\nu}\right\}^2 \leq 1 \quad (\nu = 1,2). \tag{7.29}$$

It follows from (7.29) for $\nu = 1$ and $\nu = 2$ that

$$\sum_{n=1}^{\infty}\frac{1}{2}\left\{\frac{[X(n,A,B,\alpha,p,j)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}\right\}^2(a_{p+n,1}^2 + a_{p+n,2}^2) \leq 1. \tag{7.30}$$

Therefore, we need to find the largest φ such that

$$\frac{[X(n,A,B,\varphi,p,j)]\delta(p+n,j-1)}{(B-A)(p-j+1-\varphi)\delta(p,j-1)} \leq$$

$$\frac{1}{2}\left\{\frac{[X(n,A,B,\alpha,p,j)]\delta(p+n,j-1)}{(B-A)(p-j+1-\alpha)\delta(p,j-1)}\right\}^2 \quad (n \in N) \tag{7.31}$$

that is,

$$\varphi = (p-j+1)-$$

$$\frac{2n(1+B)(B-A)(p-j+1-\alpha)^2\delta(p,j-1)}{[X(n,A,B,\alpha,p,j)]^2\delta(p+n,j-1)-2(B-A)^2(p-j+1-\alpha)^2\delta(p,j-1)}(n\in N). \tag{7.32}$$

Since

$$\Psi(n)=(p-j+1)-$$

$$\frac{2n(1+B)(B-A)(p-j+1-\alpha)^2\delta(p,j-1)}{[X(n,A,B,\alpha,p,j)]^2\delta(p+n,j-1)-2(B-A)^2(p-j+1-\alpha)^2\delta(p,j-1)}.$$

is an increasing function of $n(n\in N)$, we readily have

$$\varphi\leq\Psi(1)=(p-j+1)-$$

$$\frac{2(1+B)(B-A)(p-j+1-\alpha)^2(p-j-2)}{[X(1,A,B,\alpha,p,j)]^2(p+1)-2(B-A)^2(p-j+1-\alpha)^2(p-j-2)},$$

and Theorem 12 follows at once.

Corollary 24 . *Let the functions $f_\nu(z)$ $(\nu=1,2)$ defined by* (7.1)*be in the class*

$C(A,B,\alpha,p,j)$. Then the function $h(z)$ defined by (7.27) belongs to the class $C(A,B,$ $\xi,p,j)$ where

$$\xi=(p-j+1)-$$

$$\frac{2(1+B)(B-A)(p-j+1-\alpha)^2(p-j-1)}{[1+B+(B-A)(p-j+1-\alpha)]^2(p+1)-2(B-A)^2(p-j+1-\alpha)^2(p-j-1)}. \tag{7.33}$$

The result is sharp for the functions $f_\nu(z)(\nu=1,2)$ defined by (7.19).

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M. K. Aouf
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt.
email: [1]mkaouf127@yahoo.com,