Acta Universitatis Apulensis No. 30/2012
ISSN: 1582-5329 pp. 189-196

NEW ITERATIVE METHOD FOR SOLVING OF UNDER DETERMINED LINEAR EQUATIONS SYSTEM

Ali Shokri and Abbas Ali Shokri

Abstract. In this paper, we consider minimum norm solution of under determined orthogonal linear system equations $Ax = b$. For this purpose, after translate of system to quadratic equation $f(x) = 0$, and by use of classical Newton-Raphson rule, we calculate approximation of x.

2000 *Mathematics Subject Classification*: 65l05, 65l07, 65l20.

Key words: Minimum norm solution, Least square problems, Under determine systems.

1. Introduction

Systems of linear algebraic equations arise in all walks of life [1-5]. They represent the most basic type of system of equations and their taught to everyone as far back as 8-th grade. Yet, the complete story about linear algebraic equations is usually not taught at all. What happens when there are more equations than unknowns or fewer equations than unknowns? In this paper iterative methods for solving under determined system of linear algebraic equations that $Ax = b$ will be presented. Here A is a given $m \times n$, orthogonal matrix and b is a vector. We assume in addition that A and b are real, although this restriction in inessential in most of the methods. We like to associate this phenomenon with finding the zeros of given function f. We know that, finding the zeros of f, that is arguments x for which $f(x) = 0$, is a classical problem. In particular, determining the zeros of a polynomial (the zeros of a polynomial are also known its roots)

$$P(x) = a_0 + a_1x + \cdots + a_nx_n$$

has captured the attention of pure and applied mathematicians for centuries. However, much more general problems can be formulated in terms of finding zeros, depending upon the definition of the function $f : A \longrightarrow B$, its domain A and its range B.

For example, if $A = B = \mathbb{R}^n$, then a transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is described by n real functions $f_i(x_1, x_2, \ldots, x_n)$ of n real variables $x_1, \ldots, x_n$:

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0 \quad , \quad x^T = (x_1, \ldots, x_n)$$

The problem is solving $f(x) = 0$, becomes that of solving a system of equations

$$f_i(x_1, \ldots, x_n) = 0 \quad , \quad i = 1, 2, \ldots, n$$

Even more general problems result if A and B are linear vector spaces of infinite dimension, e.g. function spaces.

Problems of finding zeros are closely associated with problems of the the form

$$\min \quad S(x), \quad x \in \mathbb{R}^n$$

where

$$f^T(x).f(x) = \sum_{i=1}^{n} f_i^2(x) = S(x)$$

for a real function $S : R_n \longrightarrow R$ of n variables $S(x) = S(x_1, \ldots, x_n)$. For if S is differentiable and $g(x) := (\frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n})^T$ is the gradient of S, then each minimum point x of $S(x)$ is zero of the gradient $g(x) = 0$. Conversely, each zero of f is also the minimum point of some function S, where $S(x) = \|f(x)\|^2$.

2. Derivation of new method

A system of linear algebraic equations can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

Note that there are m equations with n unknowns. The number of equations and the number of unknowns can be different from one another.

Definition 2.1. *When there are fewer equations than unknowns, the system of equations is referred to as under-determined or under-constrained.*

In this case there are infinitely many possible solutions to Eq. (1).

Definition 2.2. *When the number of equations is equal to the number of unknowns, the system is referred to as uniquely determined or uniquely constrained, referring to the fact that the solution is unique.*

Definition 2.3. *When there are more equations than unknowns, the system is referred to as over-determined or over-constrained.*

In this case, there is no exact solution to the problem although approximate solutions are possible.

Using a matrix-vector notation, Eq. (1) is written as

$$Ax = b \tag{2}$$

where A is a $m \times n$ matrix, x is a $n \times 1$ vector, and b is a $m \times 1$ vector. In this paper, well assume that the equations are not linear combinations of each other. Mathematically, this implies that the rank of the matrix is the smaller of n and m, written $rank(A) = \min(n, m)$. This is called the full rank condition. By assuming that the rank of the matrix is full, it follows that the inverse of the $m \times m$ matrix $A^T A$ exists when $m \leq n$, and that the inverse of the $n \times n$ matrix AA^T exists when $n \leq m$.

When there are fewer equations than unknowns, as stated above, there are infinitely many solutions. One of the solutions that is frequently sought is the one that has the smallest norm (size). The squared norm is defined as

$$\| x \|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = x^T.x \tag{3}$$

The minimum norm problem is to minimize Eq. (3) subject to the linear algebraic equation constraints

$$f(x) = Ax - b = 0 \tag{4}$$

Let's now convert this constrained minimization problem into an unconstrained minimization problem. In this paper we consider the system (1) or (2) is under determine. Now if we define $x = (x_1, x_2, \ldots, x_n)^T$ and similarly $f_i : \mathbb{R}^n \to \mathbb{R}^m$ as belove

$$f_i(x) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i = \sum_{j=1}^{n} a_{ij}x_j - b_i \tag{5}$$

Hence by assumption $f = (f_1, \ldots, f_m)^T$, the systems of (1) or (2) becomes $f(x) = 0$ where $f : \mathbb{R}^n \to \mathbb{R}$. Now want to solve $f(x) = 0$ which x has minimum norm on the other hand we must minimize $\|f\|$ or $\|f\|^2$, moreover we have

$$\|f\|^2 = f^T(x).f(x) = \sum_{i=1}^{n} f_i^2(x) = S(x) \tag{6}$$

Then we had minimize the value of $S(x)$ where $S : \mathbb{R}^n \to \mathbb{R}$. So we must solve that $\nabla S(x) = 0$ where

$$\nabla S(x) = \begin{bmatrix} \frac{\partial S(x)}{\partial x_1} \\ \vdots \\ \frac{\partial S(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix} \tag{7}$$

Then for solving (1) and finding of its minimum norm solution, it is enough that we solve $g(x) = 0$ where

$$\nabla S(x) = g(x),$$

and $g : \mathbb{R}^n \to \mathbb{R}^n$. For this purpose we use the classical Newton -Raphson rule. The classical Newton -Raphson method is obtained by linearizing g. Linearization is also a means of constructing iterative methods to solve equation systems of the form:

$$g(x) = \begin{bmatrix} g_1(x_1, \ldots, x_n) \\ \vdots \\ g_n(x_1, \ldots, x_n) \end{bmatrix} = 0 \tag{8}$$

If we assume that $x = \xi$ is a zero for g, that x_0 is an approximation to ξ, and that g is differentiable for $x = x^0$ then to a first approximation

$$0 = g(\xi) \approx g(x^0) + Dg(x^0)(\xi - x^0) \tag{9}$$

where

$$Dg(x^0) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}, \qquad \xi - x^0 = \begin{bmatrix} \xi_1 - x_1^0 \\ \vdots \\ \xi_n - x_n^0 \end{bmatrix} \tag{10}$$

If the Jacobian $Dg(x^0)$ is nonsingular, then the equation

$$g(x^0) + Dg(x^0)(x^1 - x^0) = 0 \tag{11}$$

can be solved for x^1

$$x^1 = x^0 - (Dg(x^0))^{-1} g(x^0) \tag{12}$$

and x^1 may be taken as a closer approximation to the zero ξ. The generalization Newton method for solving systems of equation (1) is given by

$$x^{k+1} = x^k - (Dg(x^k))^{-1} g(x^k), \qquad k = 0, 1, 2, \ldots \tag{13}$$

Since

$$g_i(x) = \frac{\partial S(x)}{\partial x_i},$$

we have

$$g_i(x) = 2\sum_{i=1}^{m} a_{i1}(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i) \tag{14}$$

The Jacobian matrix $Dg(x^k)$, is now obtained by differentiating (12).

Lemma 2.4. *If $G(x)$ is the Jacobian matrix of $g(x) = 0$, then $G(x)$ is constant and symmetric $n \times n$ matrices and therefor we can assume G indicates the value $G(x)$.*

Proof. If we define $Dg(x) = G(x)$ then by a simple calculation we have

$$G_{ij} = \frac{\partial^2 s(x)}{\partial x_i \partial x_j} = 2\sum_{k=1}^{m} a_{ki}a_{kj} \tag{15}$$

And hence

$$G(x) = 2\begin{bmatrix} \sum_{k=1}^{m} a_{k1}^2 & \sum_{k=1}^{m} a_{k1}a_{k2} & \cdots & \sum_{k=1}^{m} a_{k1}a_{kn} \\ \sum_{k=1}^{m} a_{k2}a_{k1} & \sum_{k=1}^{m} a_{k2}^2 & \cdots & \sum_{k=1}^{m} a_{k2}a_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^{m} a_{kn}a_{k1} & \sum_{k=1}^{m} a_{kn}a_{k2} & \cdots & \sum_{k=1}^{m} a_{kn}^2 \end{bmatrix}_{n\times n} \tag{16}$$

Therefor $G(x)$ is independent of x, then G is constant matrix and we can use G indicated $G(x)$. Moreover by symmetric properties of

$$\sum_{k=1}^{m} a_{ki}a_{kj} = \sum_{k=1}^{m} a_{kj}a_{ki}$$

G is symmetric matrix.

Lemma 2.5. *Let A is coefficient matrix in (1). Then $G = 2A^TA$*

Proof. We know that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{(m\times n)} \tag{17}$$

Then A^TA becomes

$$A^TA = \begin{bmatrix} \sum_{k=1}^{m} a_{k1}^2 & \sum_{k=1}^{m} a_{k1}a_{k2} & \cdots & \sum_{k=1}^{m} a_{k1}a_{kn} \\ \sum_{k=1}^{m} a_{k2}a_{k1} & \sum_{k=1}^{m} a_{k2}^2 & \cdots & \sum_{k=1}^{m} a_{k2}a_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^{m} a_{kn}a_{k1} & \sum_{k=1}^{m} a_{kn}a_{k2} & \cdots & \sum_{k=1}^{m} a_{kn}^2 \end{bmatrix}_{n\times n} \tag{18}$$

Then obviously $G = 2A^TA$.

Note that the matrices A^TA and AA^T are each square, symmetric matrices. They are also positive semi-definite. To show this, let $Ax = y$ for any x. Note that $x^T(A^TA)x = y^Ty \geq 0$, so A^TA is positive semi-definite. AA^T is positive semi-definite by the same reasoning. Since, these matrices are symmetric and positive semi-definite it follows that their eigenvalues are positive or zero and their eigenvectors are real.

Theorem 2.6. *G is diagonal with non-zero entries as form*

$$G = \begin{bmatrix} 2\|A_1\|^2 & 0 & \cdots & 0 \\ 0 & 2\|A_2\|^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2\|A_n\|^2 \end{bmatrix}_{n\times n}. \tag{19}$$

Where A_i is ith column of A and moreover G is invertible and its inverse is the form

$$G^{-1} = \begin{bmatrix} \frac{1}{2\|A_1\|^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{2\|A_2\|^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{2\|A_n\|^2} \end{bmatrix}_{n\times n}. \tag{20}$$

Proof. We know that $G = 2A^TA$ so we can write

$$G = 2\begin{bmatrix} A_1^TA_1 & A_1^TA_2 & \cdots & A_1^TA_n \\ A_2^TA_1 & A_2^TA_2 & \cdots & A_2^TA_n \\ \vdots & \vdots & & \vdots \\ A_n^TA_1 & A_n^TA_2 & \cdots & A_n^TA_n \end{bmatrix}_{n\times n}. \tag{21}$$

Because orthogonality of columns of A we have $A_i^TA_j = 0$ for all $i \neq j$ and $A_i^TA_i = \|A_i\|^2$, therefor we have (19). Moreover without lose of generality all columns of A is non-zero, then

$$\|A_j\| \neq 0, \qquad \forall j = 1, 2, \cdots, n \tag{22}$$

Consequently G is invertible and the inverse of G is given by (20).

Therefor G is constant matrix and G^{-1} exist from Theorem (2.5). So whit once estimate of G^{-1}, we can use of G^{-1} in each step of our iterative method. Since it uses only once calculate G^{-1} in the initial step of our method hence can significantly reduce computational effort. Now we introduce new iterative methods as belove

$$x^{k+1} = x^k - G^{-1}g(x^k) \tag{23}$$

Then by attention to Theorem (2.5) we have

$$x^{k+1} = x^k - \begin{bmatrix} \frac{1}{2\|A_1\|^2} & & 0 & \cdots & 0 \\ 0 & \frac{1}{2\|A_2\|^2} & & \ddots & \vdots \\ \vdots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & & \frac{1}{2\|A_n\|^2} \end{bmatrix} g(x^k) \tag{24}$$

where

$$x^k = \begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{bmatrix} \tag{25}$$

Therefor

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \vdots \\ x_n^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{bmatrix} - \begin{bmatrix} \frac{g_1(x^k)}{2\|A_1\|^2} \\ \frac{g_2(x^k)}{2\|A_2\|^2} \\ \vdots \\ \frac{g_n(x^k)}{2\|A_n\|^2} \end{bmatrix} \tag{26}$$

Or equality

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \vdots \\ x_n^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^k - \frac{g_1(x^k)}{2\|A_1\|^2} \\ x_2^k - \frac{g_2(x^k)}{2\|A_2\|^2} \\ \vdots \\ x_n^k - \frac{g_n(x^k)}{2\|A_n\|^2} \end{bmatrix} \tag{27}$$

Upon choosing x^0, that is initial approximation, we can calculate x^n by (27), that is minimum norm solution of under determined of system (1). For show convergence of (27), we use the next theorem from [1].

Theorem 2.7. *Let $C \subseteq \mathbb{R}^n$ be a given open set. Further, let C_0 be a convex set with $\overline{C}_0 \subseteq C$, and let $f : C \to \mathbb{R}^n$ be a function which is differentiable for all $x \in C$. For $x^0 \in C_0$ let positive constant $r, \alpha, \beta, \gamma, h$ be given with the following properties:*

$$S_r(x^0) := \{x | \|x - x^0\| < r\} \subseteq C_0$$

and $h := \frac{\alpha\beta\gamma}{2} < 1$ and also $r := \frac{\alpha}{(1-h)}$ and let $f(x)$ have the properties

- $\forall x, y \in C_0, \quad \|Df(x) - Df(y)\| \leq \|x - y\|$
- *$Df(x)^{-1}$ exists and satisfies $\|Df(x)^{-1}\| \leq \beta, \quad \forall x \in C_0$*

- $\|Df(x^0)^{-1}f(x^0)\| \leq \alpha$

Then

1. *Beginning at* x^0*, each point*
$$x^{k+1} := x^k - Df(x^k)^{-1}f(x^k), \quad k = 0, 1, \dots \tag{28}$$
is well defined and satisfies $x^k \in S_r(x^0), \quad \forall k \geq 0.$

2. $\lim_{k\to\infty} x^k = \xi$ *exists and satisfies* $\xi \in \overline{S_r(x^0)}$ *and* $f(\xi) = 0$.

3. $\forall k \geq 0$
$$\|x^k - \xi\| \leq \alpha \frac{h^{2^k-1}}{1-h^{2^k}}. \tag{29}$$

Proof. see [1].

Theorem 2.8. *Suppose that* $g : \mathbb{R}^n \to \mathbb{R}^n$*, defined by (14). Then with start from* x^0*, (initial approximation for x), the method of (27) is well defined and convergence to root of* $g(x) = 0$*, that is minimum norm solution of system (1).*

Proof. Since G is invertible from Theorem (2.5), all conditions of previous Theorem are hold.

References

[1] Demmel, J. W., Nicholas and Higham, J., *Improved error bounds for underdetermined system solvers.* J. SIAM Matrix Anal. Appl., 14 (1), (1993), 114-139.

[2] Paige, C. C., *An error analysis of a method for solving matrix equations.* Math. Comp., 27 (122), (1973), 355-359.

[3] Peters, G., Wilkinson, J. H., *The least-squares problem and pseudo inverses*, Computer J. 13 (1970), 304-316.

[4] Serge, J. O., Lebak, J. M. and Bojanczyk, A. W., *Rank modification methods for recursive least squares problems.* Numer. Algor., 7 (2/4), (1994), 325-354.

[5] Stoer, J., Bulirsch, R., *Introduction to numerical analysis*, Springer (2002).

Ali Shokri
Faculty of Science, Department of Mathematics,
University of Maragheh,
Maragheh, Iran.
email: *shokri@maragheh.ac.ir*

Abbas Ali Shokri
Department of Mathematics, Ahar Branch, Islamic Azad University,
Ahar, Iran.
email: *a-shokri@iau-ahar.ac.ir*