Acta Universitatis Apulensis — No. 30/2012
ISSN: 1582-5329 — pp. 101-106

NEW UNIVALENCE CONDITIONS FOR SOME INTEGRAL OPERATORS

Nicoleta Ularu and Daniel Breaz

Abstract. We consider the integral operators $T_{\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n}(z)$ and $J_{\rho,\delta}(z)$. For this two operators we obtain new univalence conditions.

2000 *Mathematics Subject Classification*: Primary 30C45; Secondary 30C75.
Keywords and phrases. Analytic, univalent function class, unit disk, integral operators, Schwarz lemma.

1. Introduction and preliminaries

Let $\mathcal{A}$ the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\mathcal{U} = \{z : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.
By $\mathcal{S}$ we denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.
Pascu in [4] has proved next theorem:

Theorem 0.1. *[4] Le α be a complex number,* $\text{Re}\alpha > 0$ *and and f a regular function in $\mathcal{U}$. If*

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{2}$$

for all $z \in \mathcal{U}$, then for any complex number $\beta, \text{Re}\beta \geq \text{Re}\alpha$, the function

$$F_\beta(z) = \left(\beta\int_0^z u^{\beta-1} f'(u)du\right)^{\frac{1}{\beta}} \tag{3}$$

is in the class $\mathcal{S}$.

Theorem 0.2. *(Schwarz Lemma)[2]. Let f the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad z \in \mathcal{U}_R \tag{4}$$

the equality (in the inequality (4 for $z \neq 0$) can hold if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

The following theorem is another univalent condition which was proved by Ozaki and Nunokawa [3]:

Theorem 0.3. *[3] Let $f \in \mathcal{A}$ satisfy the following inequality:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq 1 \qquad (z \in \mathcal{U}) \tag{5}$$

then f is univalent in $\mathcal{U}$.

In [1] Al-Oboudi introduce the operator:

$$D^n f(z) = z + \sum_{k=2}^{\infty}(1 + (k-1)\varepsilon)^n a_k z^k \qquad (n \in \mathbb{N}^*)$$

with $D^0 f(0) = 0$. We will use Al-Oboudi operator to define a new operator in our paper.

2.Main Results

Theorem 0.4. *Let $\alpha \in \mathbb{C}, \mathrm{Re}\alpha > 0$ and for $i = \{1, \ldots, n\}$ we consider $\beta_i \in \mathbb{C}, \beta_i \neq 0$. Also let $M_i > 0$ and $f_i(z)$ all the functions defined by (1) that satisfies the condition (5). If*

$$|f_i(z)| \leq M_i \qquad (i \in \{1, \ldots, n\}, z \in \mathcal{U}) \tag{6}$$

and

$$\mathrm{Re}\alpha \leq \max_{1 \leq i \leq n} |\beta_i|(2M_i + 1)n \tag{7}$$

then for any complex numbers γ_i with $\mathrm{Re}\gamma_i \geq \mathrm{Re}\alpha$ the function

$$T_{\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n}(z) = \left(\sum_{i=1}^{n}\left(\frac{1}{\gamma_i} + \beta_i\right)\int_0^z u^{\sum_{i=1}^{n}\frac{1}{\gamma_i} - 1}\prod_{i=1}^{n}(f_i(u))^{\beta_i}du\right)^{\frac{1}{\sum_{i=1}^{n}\left(\frac{1}{\gamma_i} + \beta_i\right)}} \tag{8}$$

is in the univalent functions class $\mathcal{S}$.

Proof. Let g the regular function in $\mathcal{U}$ defined by

$$g(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(u)}{u} \right)^{\beta_i} du$$

From here we have that

$$\frac{g''(z)}{g'(z)} = \sum_{i=1}^{n} \beta_i \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)} \right)$$

So

$$\left| \frac{z g''(z)}{g'(z)} \right| = \left| \sum_{i=1}^{n} \beta_i \left(\frac{z f_i'(z) - f_i(z)}{f_i(z)} \right) \right| \tag{9}$$

Hence using (6) and (9) we get

$$\begin{aligned} \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{z g''(z)}{g'(z)} \right| &= \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \cdot \left| \sum_{i=1}^{n} \beta_i \left(\frac{z f_i'(z) - f_i(z)}{f_i(z)} \right) \right| \\ &\leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \cdot \sum_{i=1}^{n} |\beta_i| \left(\left| \frac{z^2 f_i'(z)}{(f_i(z))^2} \right| \frac{|f_i(z)|}{|z|} + 1 \right) \\ &\leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \cdot \sum_{i=1}^{n} |\beta_i| (2M_i + 1) \\ &\leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \cdot \max_{1 \leq i \leq n} |\beta_i| (2M_i + 1) n \end{aligned} \tag{10}$$

From (10) using the hypothesis (7) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{z g''(z)}{g'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$.
Applying Theorem 0.1 we obtain that $T_{\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n}(z)$ defined by (8) is in the univalent functions class $\mathcal{S}$. □

Corollary 0.1. *Let* $\alpha, \beta \in \mathbb{C}$*, with* $\mathrm{Re}\alpha > 0$*. Also let* $M > 0$ *and* f *the function that satisfies the inequality (5). If*

$$|f(z)| \leq M \qquad (z \in \mathcal{U})$$

and

$$\mathrm{Re}\alpha \leq |\beta|(2M + 1)$$

then for any complex number γ *with* $\mathrm{Re}\gamma \leq \mathrm{Re}\alpha$ *the function*

$$T_{\beta,\gamma} = \left(\left(\frac{1}{\gamma} + \beta \right) \int_0^z u^{\frac{1}{\gamma}-1} (f(u))^\beta du \right)^{\frac{1}{\frac{1}{\gamma}+\beta}}$$

is in the univalent function class $\mathcal{S}$.

Proof. We put $n = 1$ in Theorem 0.4 □

Theorem 0.5. *Let* $\alpha \in \mathbb{C}, \mathrm{Re}\alpha > 0$ *and* $a + bi - 1 \in \mathbb{C}$. *For* $j \in \{1, \ldots, n\}$, *let* $M_j \geq 0$ *and* $h_j \in \mathcal{A}$, $D^m h_j(z)$ *satisfies the condition (5). If*

$$|D^m h_j(z)| \leq M_j \qquad (z \in \mathcal{U}, j \in \{1, \ldots n\}) \tag{11}$$

and

$$\mathrm{Re}\alpha \leq \frac{1}{\sqrt{(a-1)^2 + b^2}} \max_{1 \leq j \leq n} (2M_j + 1)n \tag{12}$$

then for any complex numbers $\rho, \delta, \mathrm{Re}\rho\delta > \mathrm{Re}\alpha$ *the function*

$$J_{\rho,\delta}(z) = \left\{ \rho\delta \int_0^z t^{\rho\delta - 1} \prod_{j=1}^n \left(\frac{D^m h_j(t)}{t} \right)^{\frac{1}{a+bi-1}} dt \right\}^{\frac{1}{\rho\delta}} \tag{13}$$

is in the univalent function class $\mathcal{S}$.

Proof. Because $h_j \in \mathcal{A}, j \in \{1, \ldots, n\}$, from definition of $D^m h_j(z)$ we have that

$$\frac{D^m h_j(z)}{z} = 1 + \sum_{k=2}^{\infty} [1 + (k-1)\varepsilon]^m a_{k,j} z^{k-1}, m \in \mathbb{N}^*$$

and $\frac{D^m h_j(z)}{z} \neq 0$.
We consider the function

$$f(z) = \int_0^z \prod_{j=1}^n \left(\frac{D^m g_j(t)}{t} \right)^{\frac{1}{a+bi-1}} dt$$

From here we obtain that

$$\frac{zf''(z)}{f'(z)} = \frac{1}{a+bi-1} \sum_{j=1}^n \left(\frac{z(D^m g_j(t))'}{D^m g_j(t)} - 1 \right)$$

wich implies that

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| = \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\frac{1}{|a+bi-1|}\left|\sum_{j=1}^{n}\left(\frac{z(D^m g_j(t))'}{D^m g_j(t)}-1\right)\right|$$

$$\leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\frac{1}{\sqrt{(a-1)^2+b^2}}\sum_{j=1}^{n}\left(\left|\frac{z^2(D^m g_j(t))'}{(D^m g_j(t))^2}\right|\cdot\frac{|D^m g_j(t)|}{|z|}+1\right)$$

From hypothesis (11) and from Scharwz Lemma we obtain

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\frac{1}{\sqrt{(a-1)^2+b^2}}\sum_{j=1}^{n}(2M_j+1)$$

$$\leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\frac{1}{\sqrt{(a-1)^2+b^2}}\max_{1\leq j\leq n}(2M_j+1)n$$

Applying (12) we get

$$\frac{1-|z|^{2Re\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

and now from Theorem (0.1) we obtain that the function $J_{\rho,\delta}(z)$ defined by (13) belongs to $\mathcal{S}$. □

Corollary 0.2. *Let $\alpha \in \mathbb{C}, \mathrm{Re}\alpha > 0$ and $a+bi-1 \in \mathbb{C}$. For $j \in \{1,\ldots,n\}$ we consider $M_j \geq 0, h_j \in \mathcal{A}$ and $D^m h_j(z)$ satisfies the condition (5). If*

$$|D^m h_j(z)| \leq M_j \qquad (z \in \mathcal{U}, j \in \{1,\ldots n\})$$

and

$$\mathrm{Re}\alpha \leq \frac{1}{\sqrt{(a-1)^2+b^2}}\max_{1\leq j\leq n}(2M_j+1)n$$

then for any complex number $\delta, \mathrm{Re}\delta > \mathrm{Re}\alpha$ the function

$$J_\delta(z) = \left\{\delta\int_0^z t^{\delta-1}\prod_{j=1}^{n}\left(\frac{D^m h_j(t)}{t}\right)^{\frac{1}{a+bi-1}}dt\right\}^{\frac{1}{\delta}}$$

is in the univalent function class $\mathcal{S}$.

Proof. We consider $\rho = 1$ in Theorem 0.5 □

For $\rho\delta = 1$ in Theorem 0.5 we obtain

Corollary 0.3. *Let* $\alpha \in \mathbb{C}, \text{Re}\alpha > 0$ *and* $a + bi - 1 \in \mathbb{C}$. *For* $j \in \{1, \dots, n\}$ *we consider* $M_j \geq 0, h_j \in \mathcal{A}$ *and* $D^m h_j(z)$ *satisfies the condition (5). If*

$$|D^m h_j(z)| \leq M_j \qquad (z \in \mathcal{U}, j \in \{1, \dots n\})$$

and

$$\text{Re}\alpha \leq \frac{1}{\sqrt{(a-1)^2 + b^2}} \max_{1 \leq j \leq n} (2M_j + 1)n$$

and for $k \in \mathbb{N}^*$ *the function*

$$J(z) = \int_0^z \prod_{j=1}^n \left(\frac{D^m h_j(t)}{t} \right)^{\frac{1}{a+bi-1}} dt$$

is in the univalent function class $\mathcal{S}$.

Acknowledge: This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. 2004, no. 25-28, 1429-1436.

[2] O. Mayer, *The Functions Theory of One Variable Complex*, Bucureşti, 1981.

[3] Ozaki, S., Nunokawa, M., *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. 33 (1972), 392-394.

[4] N. Pascu, *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow, Brasov, (1987), 43-48.

Nicoleta Ularu
University of Piteşti
Department of Mathematics
Târgul din Vale Str., No.1, 110040, Piteşti, Argeş, România;
E-mail address: *nicoletaularuyahoo.com*

Daniel Breaz,
"1 Decembrie 1918" University of Alba Iulia,
N. Iorga Str., No. 11-13, 510009,
Alba Iulia, Alba,
Romania,
E-mail address: *dbreaz@uab.ro*