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BERTRAND MATE OF SPACELIKE BIHARMONIC CURVES WITH TIMELIKE BINORMAL ACCORDING TO FLAT METRIC IN LORENTZIAN HEISENBERG GROUP

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Abstract. In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. We characterize Bertrand mate of spacelike biharmonic curves with timelike binormal in terms of their curvature and torsion.

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1.Introduction

Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topic of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve (called Bertrand mate). Note that Bertrand mates are particular examples of offset curves used in computer-aided design.

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2} \int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [5], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2} \int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [9], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \operatorname{trace} R^N(df, \tau(f))\, df \\ &= 0, \end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study spacelike biharmonic curves with timelike binormal according to flat metric in the Lorentzian Heisenberg group Heis^3. We characterize Bertrand mate of spacelike biharmonic curves with timelike binormal in terms of their curvature and torsion.

2. Preliminaries

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = dx^2 + (xdy + dz)^2 - ((1 - x)\, dy - dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \frac{\partial}{\partial y} + (1 - x)\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 0, \quad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So we obtain that

$$R(\mathbf{e}_1, \mathbf{e}_3) = R(\mathbf{e}_1, \mathbf{e}_2) = R(\mathbf{e}_2, \mathbf{e}_3) = 0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

3. Spacelike Biharmonic Curves with Timelike Binormal According to Flat Metric in the Lorentzian Heisenberg Group Heis^3

An arbitrary curve $\gamma : I \longrightarrow Heis^3$ is spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike curve with timelike binormal and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa_1 \mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa_1 \mathbf{t} + \kappa_2 \mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= \kappa_2 \mathbf{n}, \end{aligned} \tag{3.1}$$

where κ_1, κ_2 are curvature function and torsion function, respectively and

$$\begin{aligned} g(\mathbf{t}, \mathbf{t}) &= 1, \; g(\mathbf{n}, \mathbf{n}) = 1, \; g(\mathbf{b}, \mathbf{b}) = -1, \\ g(\mathbf{t}, \mathbf{n}) &= g(\mathbf{t}, \mathbf{b}) = g(\mathbf{n}, \mathbf{b}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2 + t_3 \mathbf{e}_3, \\ \mathbf{n} &= n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + n_3 \mathbf{e}_3, \\ \mathbf{b} &= b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3. \end{aligned}$$

Theorem 3.1. *(see [11]) If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 - \kappa_2^2 &= 0, \\ \kappa_2 &= \text{constant}. \end{aligned} \tag{3.2}$$

Theorem 3.2. *(see [11]) Let $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then the parametric equations of γ are*

$$\begin{aligned} x(s) &= \cosh\varphi s + C_1, \\ y(s) &= \frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C] + \sinh[\frac{\kappa_1 s}{\sinh\varphi} + C]] + C_2, \\ z(s) &= -\frac{(-1 + C_1 + \cosh\varphi s)\sinh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\sinh\varphi} + C] \\ &\quad + \frac{\sinh^2\varphi\cosh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C] + \cosh[\frac{\kappa_1 s}{\sinh\varphi} + C]] \\ &\quad - \frac{\sinh\varphi(\cosh\varphi s + C_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\sinh\varphi} + C]] + C_3, \end{aligned} \tag{3.3}$$

where C, C_1, C_2, C_3 are constants of integration.

4. Bertrand Mate of Spacelike Biharmonic Curves According to Flat Metric in the Lorentzian Heisenberg Group Heis^3

A curve $\gamma : I \longrightarrow Heis^3$ with $\kappa_1 \neq 0$ is called a Bertrand curve if there exist a curve $\gamma_{\mathcal{B}} : I \longrightarrow Heis^3$ such that the principal normal lines of γ and $\gamma_{\mathcal{B}}$ at $s \in I$ are equal. In this case $\gamma_{\mathcal{B}}$ is called a Bertrand mate of γ.

On the other hand, let $\gamma : I \longrightarrow Heis^3$ be a Bertrand curve parametrized by arc length. A Bertrand mate of γ is as follows:

$$\gamma_{\mathcal{B}}(s) = \gamma(s) + f\mathbf{n}(s), \quad \forall s \in I, \tag{4.1}$$

where f is constant.

Lemma 4.1. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike biharmonic curve with timelike binormal according to flat metric. Then the position vector of γ is*

$$
\begin{aligned}
\gamma(s) &= (\cosh\varphi s + C_1)\mathbf{e}_1 \\
&+[(\cosh\varphi s + C_1)(\frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_2) \\
&-\frac{(-1+C_1+\cosh\varphi s)\sinh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C] \\
&+\frac{\sinh^2\varphi\cosh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]] \qquad (4.2)\\
&-\frac{\sinh\varphi(\cosh\varphi s + C_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_3]\mathbf{e}_2 \\
&[(\frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_2) \\
&-[(\cosh\varphi s + C_1)(\frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_2) \\
&-\frac{(-1+C_1+\cosh\varphi s)\sinh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C] \\
&+\frac{\sinh^2\varphi\cosh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]] \\
&-\frac{\sinh\varphi(\cosh\varphi s + C_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_3]\mathbf{e}_3,
\end{aligned}
$$

where C, C_1, C_2, C_3 are constants of integration.

Proof. Using (2.1) and (3.3), we have above system.

Theorem 4.2. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike biharmonic curve with timelike binormal and $\gamma_{\mathcal{B}}$ its Bertrand mate on $Heis^3$. Then,*

$$
\begin{aligned}
\gamma_{\mathcal{B}}(s) &= [\cosh\varphi s - \frac{f}{\kappa_1}\sinh^2\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+C] \\
&+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])+C_1)\mathbf{e}_1 \\
&+[(\cosh\varphi s + C_1)(\frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_2) \\
&-\frac{(-1+C_1+\cosh\varphi s)\sinh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]
\end{aligned}
$$

$$+\frac{\sinh^2\varphi\cosh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]] \tag{4.3}$$
$$-\frac{\sinh\varphi\,(\cosh\varphi s+C_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]$$
$$+\frac{f}{\kappa_1}(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]$$
$$+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])+C_3]\mathbf{e}_2$$

$$[(\frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_2)$$
$$-[(\cosh\varphi s+C_1)(\frac{1}{\kappa_1}\sinh^2\varphi[\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]+C_2)$$
$$-\frac{(-1+C_1+\cosh\varphi s)\sinh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]$$
$$+\frac{\sinh^2\varphi\cosh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]]$$
$$-\frac{\sinh\varphi\,(\cosh\varphi s+C_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]$$
$$+\frac{f}{\kappa_1}(\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]$$
$$-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])+C_3]\mathbf{e}_3,$$

where C, C_1, C_2, C_3 are constants of integration.

Proof. We assume that $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike biharmonic curve.

Using Lemma 4.1, we get

$$\mathbf{t}=\cosh\varphi\mathbf{e}_1+\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]\mathbf{e}_2+\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]\mathbf{e}_3. \tag{4.4}$$

Therefore, (4.4) becomes

$$\mathbf{t}=(\cosh\varphi,\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]+\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C], \tag{4.5}$$
$$(1-x)\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]-x\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]).$$

Using first equation of the system (3.2) and (2.3), we have

$$\nabla_{\mathbf{t}}\mathbf{t}=(\quad t_1'-t_2^2-t_2t_3)\mathbf{e}_1+(t_2'+t_1t_2+t_1t_3)\mathbf{e}_2$$
$$+(t_3'-t_1t_2-t_1t_3)\mathbf{e}_3.$$

On the other hand, from above equation and (3.1), we obtain

$$\begin{aligned}\nabla_{\mathbf{t}}\mathbf{t} &= -\sinh^2\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])\mathbf{e}_1 \\ &+(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C] \\ &+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])\mathbf{e}_2 \\ &+(\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C] \\ &-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])\mathbf{e}_3.\end{aligned} \tag{4.6}$$

By the use of Frenet formulas and above equation, we get

$$\begin{aligned}\mathbf{n} &= -\frac{1}{\kappa_1}\sinh^2\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+C]+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])\mathbf{e}_1 \\ &+\frac{1}{\kappa_1}(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C] \\ &+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])\mathbf{e}_2 \\ &+\frac{1}{\kappa_1}(\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+C] \\ &-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+C])\mathbf{e}_3.\end{aligned} \tag{4.7}$$

Combining (4.7) and (4.2), we obtain (4.3).

References

[1] K. Arslan, R. Ezentas, C. Murathan, T. Sasahara, *Biharmonic submanifolds 3-dimensional* (κ,μ)*-manifolds,* Internat. J. Math. Math. Sci. 22 (2005), 3575-3586.

[2] R. Caddeo, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. 12 (2001), 867–876.

[3] R. Caddeo, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds in spheres,* Israel J. Math. 130 (2002), 109–123.

[4] R. Caddeo, S. Montaldo, P. Piu, *Biharmonic curves on a surface*, Rend. Mat. Appl. 21 (2001), 143–157.

[5] J. Eells, J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109–160.

[6] J. Happel, H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media,* Prentice-Hall, New Jersey, (1965).

[7] J. Inoguchi, *Submanifolds with harmonic mean curvature in contact 3-manifolds,* Colloq. Math. 100 (2004), 163–179.

[8] G.Y. Jiang, *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A 7 (1986), 130–144.

[9] G.Y. Jiang, *2-harmonic maps and their first and second variation formulas*, Chinese Ann. Math. Ser. A 7 (1986), 389–402.

[10] J. Lopez-Bonilla, G. Ovando and J. Rivera, *Lorentz-Dirac equation and Frenet-Serret formulae*, J. Moscow Phys. Soc. 9, 83-88, 1999.

[11] T. Körpınar, E. Turhan, *On characterization spacelike biharmonic curves according to flat metric in the Lorentzian Heisenberg group* $Heis^3$*,* (submitted).

[12] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[13] S. Rahmani, *Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois*, Journal of Geometry and Physics 9 (1992), 295-302.

[14] T. Sasahara, *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors,* Publ. Math. Debrecen 67 (2005), 285–303.

[15] T. Sasahara, *Stability of biharmonic Legendre submanifolds in Sasakian space forms*, preprint.

[16 E. Turhan, *Completeness of Lorentz Metric on 3-Dimensional Heisenberg Group*, International Mathematical Forum, 3, no. 13 (2008), 639 - 644.

[17] E. Turhan, T. Körpınar, *Characterize on the Heisenberg Group with left invariant Lorentzian metric*, Demonstratio Mathematica, 2 of volume 42 (2009), 423-428

[18] E. Turhan, T. Körpınar, *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group* $Heis^3$*,* Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

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