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ON MEDIAN-PATH AND CENTRAL-PATH PROBLEMS

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Abstract. The Median-Path problem consists of locating a path on a network, minimizing a function of two parameters: accessibility to the path and total cost of the path. Applications of this problem can be found in transportation planning, water resource management and fluid transportation. The Central-Path problem is defined similarly. In this paper, we give a construction on a graph G which produces an infinite chain $G = G_0 \leq G_1 \leq G_2 \leq ...$ of graphs containing G such that for a given median (center) path P in G, P is a median (center) path in G_i for any $i \geq 1$.

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1. Introduction

Network location problems occur when one or more facilities have to be located on a network. They can be classified according to the form of the facilities, so a distinction is made between point location problems, where the facilities are to be located either in nodes or in points of the network, and path-location problems, where the facilities are path-shaped. For a complete survey on path-location problems we refer the reader to Beeker et al. (2007), Labbe et al. (1998), and Lari et al. (2008).

The Median-Path problem consists of locating a path, which minimizes a function of two parameters: the accessibility to the path and the cost of the path. Accessibility is expressed, by Buckley and Harary (1990), as the sum of the distances from the path to all the nodes not belonging to it. The cost of the path is given by the sum of the costs of the arcs belonging to the path. The Median-Path problem can therefore be defined as a bi-criterion problem, with two conflicting objective functions (the cost of the path must be increased to reduce the distance of the path and vice-versa). The complexity of the Median-Path problem on general networks is analyzed in Richey (1990) and in Hakimi et al. (1993). The problem is NP-hard on general graphs and polynomial on trees and series-parallel graphs. For some references see Minieka (1985), Minieka and Patel (1983), Morgan and Slater (1980), and Slater (1982).

Applications of the Median-Path problem arise in the design of lines (bus, underground) in a mass transportation system, where we assume that the path represents the facility and that the users demanding to reach the path are located in the nodes. The cost of the path will express the cost of setting up the facility, while the distance of the path will measure the total distance the users have to cover to reach the path.

We model the network as a graph $G = (V, E)$, where V is the vertex set with $|V| = n$ and E is the edge set with $|E| = m$. We assume that the demand points coincide with the vertices, and restrict the location of the facilities to the vertices. Each vertex v_i has a weight w_i and the edges of graph have positive lengths. We recall that the *open neighborhood* of a vertex v in a graph G is denoted by $N(v)$ or $N_G(v)$ to refer G. Thus $N(v) = \{u \in V \mid uv \in E\}$. Also for two graphs G and H by $G \leq H$ we mean that G is a subgraph of H.

We call a graph G *triangle-free* if G does not contain an triangle as an induced subgraph. We call a graph G also *claw-free* if it does not contain an star $K_{1,3}$ as an induced subgraph. Triangle-free graphs and claw-free graphs are class of well-studied graphs and play an important role in graph theory. Many of graph theory parameters deal with triangle-free graphs and claw-free graphs. To see some results on triangle-free graphs and claw-free graphs we refer the reader to for example [11]. Yet determining location problems in triangle-free graphs is open.

In this note we give a construction on a graph G which produces an infinite chain $G = G_0 \leq G_1 \leq G_2 \leq ...$ of graphs containing G such that for a given median (center) path P in G, P is a median (center) path in G_i for any $i \geq 1$. Furthermore if G is triangle-free (claw-free), then $M(G)$ is triangle-free (claw-free).

All graphs we handle in this paper are connected, and all vertices have the same weight, and also all edges have the same weight.

2. Notation and definition

Given a directed graph $G = (V, E)$, consider a weighting function $w : V \longrightarrow \Re^+ \cup \{0\}$ that associates to each vertex $v \in V$ the demand $w(v)$ observed at v, a weighting function $c : A \longrightarrow \Re^+$ that associates a length $c(a)$ to each arc $a \in E$. Given two vertices u and v, the distance $d(u, v)$ from u to v is the length of the shortest path from u to v. Let P be a path in G. The weighted distance from a vertex u to P is defined as the distance from u to that vertex in P that is the closest to u, multiplied by $w(u)$. Thus, the sum of the weighted distances from all the vertices in G to P is:

$$f(P) = \sum_{u \notin P} w(u) \min_{v \in P} d(u, v). \tag{1}$$

$f(P)$ is called the *DISTSUM* of P. If $P = \{v\}$ then we write $f(v)$ instead of $f(P)$. A path P which minimizes DISTSUM in G is called the *median path.*

Also we define the *ECCENTRICITY* of a path P by

$$E(P) = max_{v \in V}\{d(v, P)\}. \tag{2}$$

The shortest path P among those paths that minimizes ECCENTRICITY is the *central path* of G.

3. Main results

Let $G = (V, E)$ be a weighted graph (directed or undirected) with vertex set $V = \{v_1, v_2, ..., v_n\}$. For $i = 1, 2, ..., n$, let w_i be the weight of v_i. Also for $e = v_iv_j \in E$, let $w_{i,j}$ be the weight of e. We give a construction namely M-construction on G. The M-construction produce a M-graph $M(G)$ from G with $V(M(G)) = V \cup U$ where $U = \{u_1, ..., u_n\}$ and $E(M(G)) = E(G) \cup \{u_iv : v \in N_G(v_i), i = 1, ..., n\}$. The weight of new vertices and new edges (and also the direction of new edges) of $M(G)$ are as the following.

- For $i = 1, 2, ..., n$, the weight of u_i is w_i.
- For a new edge $e = u_iv_j$ the weight of e is $w_{i,j}$.
- If G is a directed graph, then for a new edge $e = u_iv_j$ the direction of e is the same direction of the edge v_iv_j, i.e. if the direction of the edge v_iv_j is $v_i \longrightarrow v_j$ then the direction of u_iv_j is $u_i \longrightarrow v_j$, and if the direction of v_iv_j is $v_j \longrightarrow v_i$ then the direction of u_iv_j is $v_j \longrightarrow u_i$.

We define the k-th M-graph of G, recursively by $M^0(G) = G$ and $M^{k+1}(G) = M(M^k(G))$ for $k \geq 0$.

Let P be a median (central) path in a graph G. We show that for any positive integer $k \geq 1$, P is a median (central) path in $M^k(G)$.

Theorem 1.*Let P be a median path in a graph G. For any positive integer $k \geq 1$, P is a median path in $M^k(G)$.*

Proof. Let G be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. For $i = 1, 2, ..., n$, let w_i be the weight of v_i, and for $e = v_iv_j \in E$, let $w_{i,j}$ be the weight of e. So $V(M(G)) = V \cup U$, where $U = \{u_1, ..., u_n\}$ and $E(M(G)) = E(G) \cup \{u_iv : v \in$

$N_G(v_i), i = 1, ..., n\}$. Also, for $i = 1, 2, ..., n$, the weight of u_i is w_i, for a new edge $e = u_i v_j$ the weight of e is $w_{i,j}$, and if G is a directed graph, then for a new edge $e = u_i v_j$ the direction e is the same direction of the edge $v_i v_j$.

Let P be a median path in G. Thus

$$f(P) = \sum_{u \notin P} w(u) \min_{v \in P} d(u, v). \tag{3}$$

is minimized. In order to refering P to the graph G, we use $f_G(P)$ instead of $f(P)$. So

$$f_G(P) = \sum_{u \in V(G) \setminus V(P)} w(u) \min_{v \in P} d(u, v). \tag{4}$$

Now in the graph $M(G)$ we have

$$\begin{aligned} f_{M(G)}(P) &= \sum_{u \in V(M(G)) \setminus V(P)} w(u) \min_{v \in P} d(u, v) \\ &= 2f_{M(G)}(P) + \sum_{v_i \in V(P)} w(u_i) \min_{v \in P} d(u_i, v). \end{aligned}$$

Let Q be a median path in $M(G)$. We show that $f_{M(G)}(Q) = f_{M(G)}(P)$. We consider two cases.

Case 1. $V(Q) \cap U = \emptyset$.
Then Q is a path in G, and $f_G(Q) \geq f_G(P)$ since P is a median path in G.

Subcase 1.1. $|V(Q)| \geq |V(P)|$. Then

$$\sum_{v_i \in V(Q)} w(u_i) \min_{v \in Q} d(u_i, v) \geq \sum_{v_i \in V(P)} w(u_i) \min_{v \in P} d(u_i, v).$$

This inequality together with $f_G(Q) \geq f_G(P)$ implies that

$$2f_{M(G)}(Q) + \sum_{v_i \in V(Q)} w(u_i) \min_{v \in Q} d(u_i, v) \geq 2f_{M(G)}(P) + \sum_{v_i \in V(P)} w(u_i) \min_{v \in P} d(u_i, v).$$

This means that $f_{M(G)}(Q) \geq f_{M(G)}(P)$. But Q is a median path in $M(G)$. Thus $f_{M(G)}(Q) = f_{M(G)}(P)$.

Subcase 1.2. $|V(Q)| < |V(P)|$.

Let $|V(Q)| = k$, where $k < |P|$. By a new labeling of the vertices of G we let $V(Q) = \{v_1^Q, v_2^Q, ..., v_k^Q\}$, where for $i = 1, 2, ..., k-1$, v_i^Q is adjacent to v_{i+1}^Q.

For $i = 1, 2, ..., k$ let $T_{v_i^Q}$ be a path with maximum number of vertices between v_i^Q and a vertex z_i in $V(G) \setminus V(Q)$ such that

$$V(T_{v_i^Q}) \cap V(Q) = \{v_1^Q, ..., v_i^Q\}.$$

Since $k < |P|$, there is an integer $j \in \{1, 2, ..., k\}$ such that the number of vertices on $T_{v_j^Q}$ between v_j^Q and z_j is greater than $\min\{j, k-j+1\}$. Without loss of generality assume that the number of vertices on $T_{v_j^Q}$ between v_j^Q and z_j is greater than j. Let $x_1, x_2, ..., x_{j-1}$ be $j-1$ vertices on $T_{v_j^Q}$ such that v_j^Q is adjacent to x_1, and x_i is adjacent to x_{i+1} for $i = 1, 2, ..., j-2$. We remove $v_1^Q, v_2^Q, ..., v_{j-1}^Q$ from Q and add $x_1, x_2, ..., x_{j-1}$ to obtain a path Q_1 in G. It follows that $f_G(Q_1) \leq f_G(Q)$ and $f_{M(G)}(Q_1) \leq f_{M(G)}(Q)$. Since Q is a median path in $M(G)$, we obtain

$$f_{M(G)}(Q_1) = f_{M(G)}(Q). \tag{5}$$

On the other hand P is a median path in G. So $f_G(Q_1) \geq f_G(P)$. Also

$$\sum_{v_i \in V(Q_1)} w(u_i) \min_{v \in Q_1} d(u_i, v) = \sum_{v_i \in V(P)} w(u_i) \min_{v \in P} d(u_i, v).$$

So

$$2f_{M(G)}(Q_1) + \sum_{v_i \in V(Q_1)} w(u_i) \min_{v \in Q_1} d(u_i, v) \geq$$

$$2f_{M(G)}(P) + \sum_{v_i \in V(P)} w(u_i) \min_{v \in P} d(u_i, v).$$

Thus $f_{M(G)}(Q_1) \geq f_{M(G)}(P)$. Now (5) implies that $f_{M(G)}(Q) \geq f_{M(G)}(P)$. But Q is a median path in $M(G)$. We conclude that $f_{M(G)}(Q) = f_{M(G)}(P)$.

Case 2. $V(Q) \cap U \neq \emptyset$. For any vertex $u_t \in V(Q) \cap U$, we replace u_t by v_t to obtain a path Q_1 in G. Now similar to the previous case, we obtain $f_{M(G)}(Q) = f_{M(G)}(P)$.

Now the result follows by an induction.

Theorem 2. *Let P be a central path in a graph G. For any positive integer $k \geq 1$, P is a central path in $M^k(G)$.*

The proof of Theorem 2 is similar to the proof of Theorem 1, and therefore is omitted.

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