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AN ITERATIVE METHOD FOR THE GENERALIZED CENTRO-SYMMETRIC SOLUTION OF A LINEAR MATRIX EQUATION $AXB + CYD = E$

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Abstract. A matrix $P \in R^{n\times n}$ is said to be a symmetric orthogonal matrix if $P = P^T = P^{-1}$. A matrix $A \in R^{n\times n}$ is said to be generalized centro-symmetric (generalized central anti-symmetric)with respect to P, if $A = PAP(A = -PAP)$. In this paper, an iterative method is constructed to solve the generalized centro-symmetric solutions of a linear matrix equation $AXB + CYD = E$, with real pair matrices X and Y. We show when the matrix equation is consistent over generalized centro-symmetric pair matrices X and Y, for any initial pair matrices X_0 and Y_0, the generalized centro-symmetric solution can be obtained within finite iterative steps in the absence of roundoff errors, and the minimum norm of the generalized centro-symmetric solutions can be obtained by choosing a special kind of initial pair matrices. Furthermore, the optimal approximation pair solution $\widehat{X}$ and $\widehat{Y}$ to a given matrices $\overline{X}$ and $\overline{Y}$ can be derived.

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1. Introduction

Throughout the paper, the notations $R^{n\times n}$, $SOR^{n\times n}$ represent the set of all $n \times n$ real and real symmetric orthogonal matrices, respectively. $A \otimes B$ stands for the Kronecker product of matrices A and B. A^T, $trace(A)$ and $R(A)$ denote the transpose, trace and column space of the matrix A respectively. Also $\mathrm{vec}(A)$ represents the vector operator. i.e. $\mathrm{vec}(A) = (a_1^T, \cdots, a_n^T)^T \in R^{mn}$ for the matrix $A = (a_1, \cdots, a_n) \in R^{m\times n}$, $a_i \in R^m, i = 1, \cdots, n$. We define an inner product as $\langle A, B\rangle = trace(B^T A)$. Then the norm of a matrix A generated by this inner product is the Frobenius norm and is denoted by $\|A\|$.

Definition 1.1. *Let P be some real symmetric orthogonal $n \times n$ matrix, i.e. $P = P^T = P^{-1}$. If $A = PAP$, then A is called a generalized centro-symmetric* *matrix with respect to P. $CSR_P^{n\times n}$ denotes the set of order n generalized centro-symmetric matrices with respect to $P \in SOR^{n\times n}$.*

The following two problems are considered in this paper.

Problem I. *Given matrices $A \in R^{p\times n}, B \in R^{n\times q}, C \in \mathcal{R}^{p\times m}, D \in R^{m\times q}$, and $E \in R^{p\times q}$, find a pair matrices $X \in CSR_P^{n\times n}$ and $Y \in CSR_P^{m\times m}$, such that*

$$AXB + CYD = E. \tag{1}$$

Problem II. *When Problem I is consistent, let S_E denote the set of solutions of Problem I*

$$S_E = \{(X,Y) \in CSR_P^{n\times n} \times CSR_P^{m\times m} : AXB + CYD = E\}.$$

For given pair matrices $\overline{X} \in R^{n\times n}$ and $\overline{Y} \in R^{m\times m}$,find the generalized centro-symmetric pair matrices $\widehat{X} \in S_E$ and $\widehat{Y} \in S_E$ such that

$$\|\widehat{X} - \overline{X}\| + \|\widehat{Y} - \overline{Y}\| = \min_{(X,Y)\in S_E} \{\|\widehat{X} - \overline{X}\| + \|\widehat{Y} - \overline{Y}\|\}. \tag{2}$$

Matrix equation is one of the topics of very active research in computational mathematics, and has been widely applied in various areas. Many results have been obtained about equation (1).For example, Chu [1] gave the consistency conditions and the minmum norm solution by making use of the generalized singular value decomposition(GSVD). Huang and Zeng [2] and Özgüler [3], respectively, gave the solvability conditions over a simple Artinian ring and principal ideal domain by using the generalized inverse. Shim and Chen [4],Xu, Wei and Zheng [5] presented the least square solution with the minimun norm by using the canonical correlation decomposition(CCD) and GSVD. Yuan,Liao and Lei [6] obtain a unique least squares symmetric the Kronecker product of matrices. Peng and Peng [7] solve the solution of equation (1) by using iterative method Sheng and Chen [8] obtained the symmetric and skew symmetric of equation (1) by using iterative method. In this paper, we will use iterative method to solve the generalized centro-symmetric solutions of equation (1).

This paper is organized as follows: In Section 2, we introduce an iterative algorithm for solving Problem I. Then we prove several properties of Algorithm I. Also, when the linear matrix equation (1) is consistent, we show for any initial generalized centro-symmetric matrix pair X_0 and Y_0, a generalized centro-symmetric solution can be obtained within finite iteration steps, and also show that if the initial matrix is chosen as $X_0 = A^THB^T + PA^THB^TP$ and $Y_0 = C^THD^T + PC^THD^TP$, where

H is arbitrary, then the generalized centro-symmetric X^* and Y^* obtained by the iterative method is the minimum norm solution. The optimal approximation generalized centro-symmetric solution to given generalized centro-symmetric matrix pair $\overline{\overline{X}}$ and $\overline{\overline{Y}}$ in the solution set of the linear matrix equation (1) is obtained in Section 3.

2.The iterative algorithm for solving Problem I

In this section, we will construct an iterative method to solve Problem I. Then, some basic properties of the introduced iterative method are described. Finally, we show that it is convergent.

Algorithm I.
Step 1. Input matrices A, B, C, D, E and $X_0 \in CSR_P^{n\times n}$, $Y_0 \in CSR_P^{m\times m}$.
Step 2.

$$\begin{aligned} R_0 &= E - AX_0B - CY_0D; \\ P_0 &= A^T R_0 B^T; \\ P_0^s &= \frac{P_0 + PP_0P}{2}; \\ Q_0 &= C^T R_0 D^T; \\ Q_0^s &= \frac{Q_0 + PQ_0P}{2}; \\ k &= 0. \end{aligned}$$

Step 3. If $R_k = 0$, then stop; else, $k = k + 1$.
Step 4. Calculate

$$\begin{aligned} X_k &= X_{k-1} + \frac{\|R_{k-1}\|^2}{\|P_{k-1}^s\|^2 + \|Q_{k-1}^s\|^2} P_{k-1}^s; \\ Y_k &= Y_{k-1} + \frac{\|R_{k-1}\|^2}{\|P_{k-1}^s\|^2 + \|Q_{k-1}^s\|^2} Q_{k-1}^s; \\ R_k &= E - AX_kB - CY_kD \\ &= R_{k-1} - \frac{\|R_{k-1}\|^2}{\|P_{k-1}^s\|^2 + \|Q_{k-1}^s\|^2}(AP_{k-1}^sB + CQ_{k-1}^sD); \end{aligned}$$

$$\begin{aligned}
P_k &= A^T R_k B^T; \\
Q_k &= C^T R_k D^T; \\
P_k^s &= \frac{P_k + P P_k P}{2} - \frac{tr(P_k P_{k-1}^s) + tr(Q_k Q_{k-1}^s)}{\|P_{k-1}^s\|^2 + \|Q_{k-1}^s\|^2} P_{k-1}^s; \\
Q_k^s &= \frac{Q_k + P Q_k P}{2} - \frac{tr(P_k P_{k-1}^s) + tr(Q_k Q_{k-1}^s)}{\|P_{k-1}^s\|^2 + \|Q_{k-1}^s\|^2} Q_{k-1}^s.
\end{aligned}$$

Step 5. Go to step 3.

Remark 2.1 Obviously, $P_k^s, Q_k^s \in CSR_p^{n\times n}$ and $X_k \in CSR_P^{n\times n}$,$Y_k \in CSR_P^{m\times m}$ for $k = 0, 1, \cdots$, from Algorithm I.

Lemma 2.2[8] *Let $A, B \in R^{n\times n}$, then we have*

$$\langle A, B\rangle = \langle B, A\rangle = \langle A^T, B^T\rangle = \langle B^T, A^T\rangle.$$

Lemma 2.3 *Let $P \in SOR^{n\times n}$. i.e.$P = P^T = P^{-1}$, $A \in R^{n\times n}$,$B \in CSR_P^{n\times n}$, then*

$$\langle \frac{A + PAP}{2}, B\rangle = \langle A, B\rangle.$$

Proof.

$$\begin{aligned}
\langle A, B\rangle &= tr(B^T A) = tr[B^T(\frac{A + PAP}{2} + \frac{A - PAP}{2})] \\
&= tr[\frac{B^T(A + PAP)}{2}] + tr(\frac{B^T A}{2}) - tr(\frac{B^T PAP}{2}) \\
&= \langle \frac{A + PAP}{2}, B\rangle + tr(\frac{B^T A}{2}) - tr(\frac{P^T B^T P^T A}{2}) \\
&= \langle \frac{A + PAP}{2}, B\rangle + tr(\frac{B^T A}{2}) - tr(\frac{B^T A}{2}) \\
&= \langle \frac{A + PAP}{2}, B\rangle.
\end{aligned}$$

Lemma 2.4 *Assume that the linear matrix equation* (1) *is consistent and* (X^*, Y^*) *is one of its solutions, then, for any initial generalized centro-symmetric matrix pair* (X_0, Y_0)*, the sequences* $\{X_i\}, \{Y_i\}, \{R_i\}, \{P_i^s\}$ *and* $\{Q_i^s\}$ *generalized by Algorithm I satisfy*

$$\langle P_i^s, X^* - X_i\rangle + \langle Q_i^s, Y^* - Y_i\rangle = \|R_i\|^2, (i = 0, 1, 2, \cdots). \tag{3}$$

Proof. We prove the conclusion by induction and notice that X^*, X_0, Y^*, Y_0 are all generalized centro-symmetric matrices. When $i = 0$, we have

$$\begin{aligned}
\langle P_0^s, X^* - X_0\rangle + \langle Q_0^s, Y^* - Y_0\rangle &= \langle \frac{P_0 + PP_0P}{2}, X^* - X_0\rangle + \langle \frac{Q_0 + PQ_0P}{2}, Y^* - Y_0\rangle \\
&= \langle P_0, X^* - X_0\rangle + \langle Q_0, Y^* - Y_0\rangle \\
&= \langle A^T R_0 B^T, X^* - X_0\rangle + \langle C^T R_0 D^T, Y^* - Y_0\rangle \\
&= \langle R_0, A(X^* - X_0)B\rangle + \langle R_0, C(Y^* - Y_0)D\rangle \\
&= \langle R_0, A(X^* - X_0)B + C(Y^* - Y_0)D\rangle \\
&= \langle R_0, R_0\rangle \\
&= \|R_0\|^2.
\end{aligned}$$

Assume that (3) holds for $i = t(for\ t \geqslant 0)$, that is $\langle P_t^s, X^* - X_t\rangle + \langle Q_t^s, Y^* - Y_t\rangle = \|R_t\|^2$, then for $i = t + 1$, we have

$$\begin{aligned}
&\langle P_{t+1}^s, X^* - X_{t+1}\rangle + \langle Q_{t+1}^s, Y^* - Y_{t+1}\rangle \\
&= \langle \frac{P_{t+1} + PP_{t+1}P}{2} - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2} P_t^s, X^* - X_{t+1}\rangle \\
&\quad + \langle \frac{Q_{t+1} + PQ_{t+1}P}{2} - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2} Q_t^s, Y^* - Y_{t+1}\rangle \\
&= \langle \frac{P_{t+1} + PP_{t+1}P}{2}, X^* - X_{t+1}\rangle + \langle \frac{Q_{t+1} + PQ_{t+1}P}{2}, Y^* - Y_{t+1}\rangle \\
&\quad - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2} [\langle P_t^s, X^* - X_{t+1}\rangle + \langle Q_t^s, Y^* - Y_{t+1}\rangle] \\
&= \langle P_{t+1}, X^* - X_{t+1}\rangle + \langle Q_{t+1}, Y^* - Y_{t+1}\rangle - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2} \langle P_t^s, X^* - X_{t+1}\rangle \\
&\quad - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2} \langle Q_t^s, Y^* - Y_{t+1}\rangle \\
&= \langle A^T R_{t+1} B^T, X^* - X_{t+1}\rangle + \langle C^T R_{t+1} D^T, Y^* - Y_{t+1}\rangle \\
&\quad - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2} \langle P_t^s, X^* - X_t - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2} P_t^s\rangle \\
&\quad - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2} \langle Q_t^s, Y^* - Y_t - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2} Q_t^s\rangle
\end{aligned}$$

$$= \|R_{t+1}\|^2 - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle P_t^s, X^* - X_t\rangle - \langle P_t^s, \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}P_t^s\rangle]$$
$$- \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle Q_t^s, Y^* - Y_t\rangle - \langle Q_t^s, \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}Q_t^s\rangle]$$
$$= \|R_{t+1}\|^2 - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2}\|R_t\|^2$$
$$+ \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2}\frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle P_t^s, P_t^s\rangle + \langle Q_t^s, Q_t^s\rangle]$$
$$= \|R_{t+1}\|^2.$$

By the principle of induction, the conclusion (3) holds for all $i = 0, 1, 2, \cdots$.

Remark 2.5 From the formulae of P_i^s and Q_i^s i in Algorithm I and Lemma 2.4, we know that if the linear matrix equation (1) is consistent, then, $R_i = 0$ if and only if $P_i = 0$ and $Q_i = 0$. This result implies that if there exists a positive number k such that $P_k^s = Q_k^s = 0$ but $R_k \neq 0$, then the linear matrix equation (1) has no generalized centro-symmetric solution. Hence, the solvability of the linear matrix equation (1) can be determined automatically by Algorithm I in the absence of roundoff errors.

Lemma 2.6 *Assume that the linear matrix equation* (1) *is consistent and the sequences* $\{R_i\}$, $\{P_i^s\}$ *and* $\{Q_i^s\}$, *where* $\|R_i\|^2 \neq 0 (i = 0, 1, 2, \cdots, k)$ *generated by Algorithm I. Then we have*

$$\langle R_i, R_j\rangle = 0, \ \langle P_i^s, P_j^s\rangle + \langle Q_i^s, Q_j^s\rangle = 0, (i \neq j, i, j = 0, 1, \cdots, k) \tag{4}$$

Proof. From Lemma 2.2 we know that $\langle A, B\rangle = \langle B, A\rangle$ holds for all matrices A and B in $R^{p\times q}$, we only prove that the conclusion holds for all $0 \leqslant i < j \leqslant k$. Using induction and two steps are required.

Step 1. Show that $\langle R_i, R_{i+1}\rangle = 0$ and $\langle P_i^s, P_{i+1}^s\rangle + \langle Q_i^s, Q_{i+1}^s\rangle = 0$ for all $i = 0, 1, 2, \cdots, k$. To prove this conclusion, we also use induction.

For $i = 0$, we have

$$\begin{aligned}
\langle R_0, R_1\rangle &= \langle R_0, R_0 - \frac{\|R_0\|^2}{\|P_0^s\|^2 + \|Q_0^s\|^2}(AP_0^sB + CQ_0^sD)\rangle \\
&= \|R_0\|^2 - \frac{\|R_0\|^2}{\|P_0^s\|^2 + \|Q_0^s\|^2}[\langle R_0, AP_0^sB\rangle + \langle R_0, CQ_0^sD\rangle] \\
&= \|R_0\|^2 - \frac{\|R_0\|^2}{\|P_0^s\|^2 + \|Q_0^s\|^2}[\langle P_0, P_0^s\rangle + \langle Q_0, Q_0^s\rangle] \\
&= \|R_0\|^2 - \frac{\|R_0\|^2}{\|P_0^s\|^2 + \|Q_0^s\|^2}(\|P_0^s\|^2 + \|Q_0^s\|^2) \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
\langle P_0^s, P_1^s\rangle + \langle Q_0^s, Q_1^s\rangle &= \langle P_0^s, \frac{P_1 + PP_1P}{2} - \frac{tr(P_1P_0^s) + tr(Q_1Q_0^s)}{\|P_0^s\|^2 + \|Q_0^s\|^2}P_0^s\rangle \\
&\quad + \langle Q_0^s, \frac{Q_1 + PQ_1P}{2} - \frac{tr(P_1P_0^s) + tr(Q_1Q_0^s)}{\|P_0^s\|^2 + \|Q_0^s\|^2}Q_0^s\rangle \\
&= \langle P_0^s, \frac{P_1 + PP_1P}{2}\rangle + \langle Q_0^s, \frac{Q_1 + PQ_1P}{2}\rangle \\
&\quad - \frac{tr(P_1P_0^s) + tr(Q_1Q_0^s)}{\|P_0^s\|^2 + \|Q_0^s\|^2}[\langle P_0^s, P_0^s\rangle + \langle Q_0^s, Q_0^s\rangle] \\
&= tr(P_1P_0^s) + tr(Q_1Q_0^s) - [tr(P_1P_0^s) + tr(Q_1Q_0^s)] \\
&= 0.
\end{aligned}$$

Assume (4) holds for all $i \leqslant t(\text{for} 0 < t < k)$, then

$$\begin{aligned}
\langle R_t, R_{t+1}\rangle &= \langle R_t, R_t - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}(AP_t^sB + CQ_t^sD)\rangle \\
&= \|R_t\|^2 - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle A^TR_tB^T, P_t^s\rangle + \langle C^TR_tD^T, Q_t^s\rangle] \\
&= \|R_t\|^2 - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle P_t, P_t^s\rangle + \langle Q_t, Q_t^s\rangle] \\
&= \|R_t\|^2 - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle \frac{P_t + PP_tP}{2}, P_t^s\rangle + \langle \frac{Q_t + PQ_tP}{2}, Q_t^s\rangle]
\end{aligned}$$

$$
\begin{aligned}
&= \|R_t\|^2 - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}\langle \frac{P_t + PP_tP}{2} - \frac{tr(P_tP_{t-1}^s) + tr(Q_tQ_{t-1}^s)}{\|P_{t-1}^s\|^2 + \|Q_{t-1}^s\|^2}P_{t-1}^s, P_t^s\rangle \\
&\qquad - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}\langle \frac{Q_t + PQ_tP}{2} - \frac{tr(P_tP_{t-1}^s) + tr(Q_tQ_{t-1}^s)}{\|P_{t-1}^s\|^2 + \|Q_{t-1}^s\|^2}Q_{t-1}^s, Q_t^s\rangle \\
&= \|R_t\|^2 - \frac{\|R_t\|^2}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle P_t^s, P_t^s\rangle + \langle Q_t^s, Q_t^s\rangle] \\
&= \|R_t\|^2 - \|R_t\|^2 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\langle P_t^s, P_{t+1}^s\rangle + \langle Q_t^s, Q_{t+1}^s\rangle &= \langle P_t^s, \frac{P_{t+1} + PP_{t+1}P}{2} - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2}P_t^s\rangle \\
&\quad + \langle Q_t^s, \frac{Q_{t+1} + PQ_{t+1}P}{2} - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2}Q_t^s\rangle \\
&= \langle P_t^s, \frac{P_{t+1} + PP_{t+1}P}{2}\rangle + \langle Q_t^s, \frac{Q_{t+1} + PQ_{t+1}P}{2}\rangle \\
&\quad - \frac{tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)}{\|P_t^s\|^2 + \|Q_t^s\|^2}[\langle P_t^s, P_t^s\rangle + \langle Q_t^s, Q_t^s\rangle] \\
&= tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s) - [tr(P_{t+1}P_t^s) + tr(Q_{t+1}Q_t^s)] \\
&= 0.
\end{aligned}
$$

By the principle of induction, $\langle R_i, R_{i+1}\rangle = 0$ and $\langle P_i^s, P_{i+1}^s\rangle + \langle Q_i^s, Q_{i+1}^s\rangle = 0$ hold for all $i = 0, 1, \cdots, k$.

Step 2. Assume that $\langle R_i, R_{i+l}\rangle = 0$ and $\langle P_i^s, P_{i+l}^s\rangle + \langle Q_i^s, Q_{i+l}^s\rangle = 0$ hold for all $0 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant k$, show that $\langle R_i, R_{i+l+1}\rangle = 0$ and $\langle P_i^s, P_{i+l+1}^s\rangle + \langle Q_i^s, Q_{i+l+1}^s\rangle = 0$.

$$
\begin{aligned}
\langle R_i, R_{i+l+1}\rangle &= \langle R_i, R_{i+l} - \frac{\|R_{i+l}\|^2}{\|P_{i+l}^s\|^2 + \|Q_{i+l}^s\|^2}(AP_{i+l}^sB + CQ_{i+l}^sD)\rangle \\
&= \langle R_i, R_{i+l}\rangle - \frac{\|R_{i+l}\|^2}{\|P_{i+l}^s\|^2 + \|Q_{i+l}^s\|^2}[\langle R_i, AP_{i+l}^sB\rangle + \langle R_i, CP_{i+l}^sD\rangle] \\
&= -\frac{\|R_{i+l}\|^2}{\|P_{i+l}^s\|^2 + \|Q_{i+l}^s\|^2}[\langle P_i, P_{i+l}^s\rangle + \langle Q_i, Q_{i+l}^s\rangle] \\
&= -\frac{\|R_{i+l}\|^2}{\|P_{i+l}^s\|^2 + \|Q_{i+l}^s\|^2}[\langle \frac{P_i + PP_iP}{2}, P_{i+l}^s\rangle + \langle \frac{Q_i + PQ_iP}{2}, Q_{i+l}^s\rangle] \\
&= -\frac{\|R_{i+l}\|^2}{\|P_{i+l}^s\|^2 + \|Q_{i+l}^s\|^2}\langle \frac{P_i + PP_iP}{2} - \frac{tr(P_iP_{i-1}^s) + tr(Q_iQ_{i-1}^s)}{\|P_{i-1}^s\|^2 + \|Q_{i-1}^s\|^2}P_{i-1}^s, P_{i+l}^s\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\|R_{i+l}\|^2}{\|P^s_{i+l}\|^2 + \|Q^s_{i+l}\|^2}\langle\frac{Q_i + PQ_iP}{2} - \frac{tr(P_iP^s_{i-1}) + tr(Q_iQ^s_{i-1})}{\|P^s_{i-1}\|^2 + \|Q^s_{i-1}\|^2}Q^s_{i-1}, Q^s_{i+l}\rangle \\
&= -\frac{\|R_{i+l}\|^2}{\|P^s_{i+l}\|^2 + \|Q^s_{i+l}\|^2}[\langle P^s_i, P^s_{i+l}\rangle + \langle Q^s_i, Q^s_{i+l}\rangle] \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle P^s_i, P^s_{i+l+1}\rangle + \langle Q^s_i, Q^s_{i+l+1}\rangle \\
&= \langle P^s_i, \frac{P_{i+l+1} + PP_{i+l+1}P}{2} - \frac{tr(P_{i+l+1}P^s_{i+l}) + tr(Q_{i+l+1}Q^s_{i+l})}{\|P^s_{i+l}\|^2 + \|Q^s_{i+l}\|^2}P^s_{i+l}\rangle \\
&\quad + \langle Q^s_i, \frac{Q_{i+l+1} + PQ_{i+l+1}P}{2} - \frac{tr(P_{i+l+1}P^s_{i+l}) + tr(Q_{i+l+1}Q^s_{i+l})}{\|P^s_{i+l}\|^2 + \|Q^s_{i+l}\|^2}Q^s_{i+l}\rangle \\
&= \langle P^s_i, \frac{P_{i+l+1} + PP_{i+l+1}P}{2}\rangle + \langle Q^s_i, \frac{Q_{i+l+1} + PQ_{i+l+1}P}{2}\rangle \\
&\quad - \frac{tr(P_{i+l+1}P^s_{i+l}) + tr(Q_{i+l+1}Q^s_{i+l})}{\|P^s_{i+l}\|^2 + \|Q^s_{i+l}\|^2}[\langle P^s_i, P^s_{i+l}\rangle + \langle Q^s_i, Q^s_{i+l}\rangle] \\
&= \langle P^s_i, \frac{P_{i+l+1} + PP_{i+l+1}P}{2}\rangle + \langle Q^s_i, \frac{Q_{i+l+1} + PQ_{i+l+1}P}{2}\rangle \\
&= \langle P^s_i, P_{i+l+1}\rangle + \langle Q^s_i, Q_{i+l+1}\rangle \\
&= \langle P^s_i, A^TR_{i+l+1}B^T\rangle + \langle Q^s_i, C^TR_{i+l+1}D^T\rangle \\
&= \langle AP^s_iB, R_{i+l+1}\rangle + \langle CQ^s_iD, R_{i+l+1}\rangle \\
&= \frac{\|P^s_i\|^2 + \|Q^s_i\|^2}{\|R^s_i\|^2}[\langle A(X_{i+1} - X_i)B, R_{i+l+1}\rangle + \langle C(X_{i+1} - X_i)D, R_{i+l+1}\rangle] \\
&= \frac{\|P^s_i\|^2 + \|Q^s_i\|^2}{\|R^s_i\|^2}[\langle R_i - R_{i+1}, R_{i+l+1}\rangle] \\
&= 0.
\end{aligned}
$$

From Step 1 and Step 2, the conclusion (4) holds by the principle of induction.

Remark 2.7 Lemma 2.6 implies that if the linear matrix equation (1) is consistent, then, for any initial generalized centro-symmetric matrix pair X_0 and Y_0, a solution can be obtained within at most pq iteration steps. Since the $R_0, R_1, \cdots$ are orthogonal each other in the finite dimension matrix space $R^{p\times q}$, it is certain that there exists a positive number $k \leqslant pq$ such that $R_k = 0$.

Lemma 2.8 [9] *Let the linear system $Ax = b$ be consistent, if x^* is a solution, satisfied $x^* \in R(A^*)$, then x^* is the unique minimum normal solution of it.*

Lemma 2.9 *In Algorithm I, if we choose the* $X_0 = A^T H B^T + P A^T H B^T P$ *and* $Y_0 = C^T H D^T + P C^T H D^T P$, *where* H *is an arbitrary matrix in* $R^{p\times q}$, *then the sequences of* $\{X_k\}$ *and* $\{Y_k\}$ *generated by Algorithm I have the following properties* $X_k = A^T \widetilde{H}_k B^T + P A^T \widetilde{H}_k^T B^T P$ *and* $Y_k = A^T \widehat{H}_k B^T + P A^T \widehat{H}_k^T B^T P$, *where* $\widetilde{H}_k$ *and* $\widehat{H}_k$ *is some matrix in* $R^{p\times q}$.

Consider the following system of matrix equations

$$\begin{cases} AXB + CYD = E \\ APXPB + CPYPD = E \end{cases} \tag{5}$$

Obviously, the solvability of the above system of matrix equation is equivalent to Problem I. The system of matrix equations (5) is equivalent to

$$\begin{pmatrix} B^T \bigotimes A & D^T \bigotimes C \\ B^T P \bigotimes AP & D^T P \bigotimes CP \end{pmatrix} \begin{pmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}(E) \\ \mathrm{vec}(E) \end{pmatrix}.$$

Now suppose $H \in R^{n\times n}$ is obviously matrices, we have

$$\begin{aligned} \mathrm{vec}\begin{pmatrix} A^T H B^T + P A^T H B^T P \\ C^T H D^T + P C^T H D^T P \end{pmatrix} &= \begin{pmatrix} B \bigotimes A^T & PB \bigotimes PA^T \\ D \bigotimes C^T & PD \bigotimes PC^T \end{pmatrix} \begin{pmatrix} \mathrm{vec}(H) \\ \mathrm{vec}(H) \end{pmatrix} \\ &= \begin{pmatrix} B^T \bigotimes A & D^T \bigotimes C \\ B^T P \bigotimes AP & D^T P \bigotimes CP \end{pmatrix}^T \begin{pmatrix} \mathrm{vec}(H) \\ \mathrm{vec}(H) \end{pmatrix} \\ &\in R\left(\begin{pmatrix} B^T \bigotimes A & D^T \bigotimes C \\ B^T P \bigotimes AP & D^T P \bigotimes CP \end{pmatrix}^T\right) \end{aligned}$$

Obviously, if we consider

$$X_0 = A^T H B^T + P A^T H B^T P, Y_0 = C^T H D^T + P C^T H D^T P,$$

then all Y_k, generated by Algorithm I satisfy

$$\begin{pmatrix} \mathrm{vec}(X_k) \\ \mathrm{vec}(Y_k) \end{pmatrix} \in R\left(\begin{pmatrix} B^T \bigotimes A & D^T \bigotimes C \\ B^T P \bigotimes AP & D^T P \bigotimes CP \end{pmatrix}^T\right).$$

Hence by Lemma 2.8, if we take an initial matrices

$$X_0 = A^T H B^T + P A^T H B^T P, Y_0 = C^T H D^T + P C^T H D^T P,$$

then X^* and Y^* generated by Algorithm I are the least Frobenius norm generalized centro-symmetric solution. By using the above conclusions, we can prove the following theorem.

Theorem 2.10 *Suppose that Problem I is consistent. If we take initial matrices*

$$X_0 = A^T HB^T + PA^T HB^T P, Y_0 = C^T HD^T + PC^T HD^T P,$$

where $H \in R^{n\times n}$ is arbitrary, or more especially X_0,Y_0, then the solutions X^* and Y^* are the least Frobenius norm generalized centro-symmetric solution of Problem I.

3.The Solution of Problem II

We assume that $\overline{X} \in R^{n\times n}$ and $\overline{Y} \in R^{m\times m}$ in Problem II, it is well known that a generalized centro-symmetric matrix and a generalized central anti-symmetric are orthogonal each other, for any $X \in CSR_P^{n\times n}$ and $Y \in CSR_P^{m\times m}$, we have that

$$\begin{aligned} &\|X-\overline{X}\|^2+\|Y-\overline{Y}\|^2 \\ &=\|X-(\frac{\overline{X}+P\overline{X}P}{2}+\frac{\overline{X}-P\overline{X}P}{2})\|^2+\|Y-(\frac{\overline{Y}+P\overline{Y}P}{2}+\frac{\overline{Y}-P\overline{Y}P}{2})\|^2 \\ &=\|X-\frac{\overline{X}+P\overline{X}P}{2}\|^2+\|\frac{\overline{X}-P\overline{X}P}{2}\|^2+\|Y-\frac{\overline{Y}+P\overline{Y}P}{2}\|^2+\|\frac{\overline{Y}-P\overline{Y}P}{2}\|^2. \end{aligned}$$

Denote $\overline{\overline{X}} = \frac{\overline{X}+P\overline{X}P}{2}$ and $\overline{\overline{Y}} = \frac{\overline{Y}+P\overline{Y}P}{2}$, when the linear matrix equation (1) is consistent, the solution set S_E of the matrix equation (1) is no-empty, then

$$AXB + CYD = E \Leftrightarrow A(X-\overline{\overline{X}})B + C(Y-\overline{\overline{Y}})D = E - A\overline{\overline{X}}B - C\overline{\overline{Y}}D.$$

Let $\widetilde{X} = X-\overline{\overline{X}}, \widetilde{Y} = Y-\overline{\overline{Y}}$ and $\widetilde{E} = E - A\overline{\overline{X}}B - C\overline{\overline{Y}}D$, then the matrix nearness Problem II is the equivalent to find the minimum norm solution of the pair of matrix equations

$$A\widetilde{X}B + C\widetilde{X}D = \widetilde{E}. \tag{6}$$

By using Algorithm I, and let the initial matrix $\widetilde{X}_0 = A^T HB^T + PA^T HB^T P, \widetilde{Y}_0 = C^T HD^T + PC^T HD^T P$, where H is an arbitrary matrix in $R^{p\times q}$, more specially, let $\widetilde{X}_0 = 0$ and $\widetilde{Y}_0 = 0$, we can obtain the unique minimum norm solution $\widetilde{X}^*$ and $\widetilde{Y}^*$ of linear matrix equation (6). Once the above matrix $\widetilde{X}^*$ and $\widetilde{Y}^*$ are obtained, the unique generalized centro-symmetric solution pair of the matrix nearness Problem II can be obtained. In this case, $\widehat{X}$ and $\widehat{Y}$ can be expressed $\widehat{X} = \widetilde{X}^* + \overline{\overline{X}} = \widetilde{X}^* + \frac{\overline{X}+P\overline{X}P}{2}$ and $\widehat{Y} = \widetilde{Y}^* + \overline{\overline{Y}} = \widetilde{Y}^* + \frac{\overline{Y}+P\overline{Y}P}{2}$, respectively.

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