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SUFFICIENT CONDITIONS FOR UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper we study certain integral operators and we determine conditions for their univalence using some univalence criteria obtained by Ahlfors [1], Becker [2] and Pascu [4].

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1. Introduction and preliminaries

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\},$$

$$A_n = \{f \in \mathcal{H} : \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\},$$

with $A_1 = A$,

$$S = \{f \in A \mid f \text{ is univalent in } U\}.$$

In order to prove our main results we shall make use of the following lemmas.

Lemma A. ([1], [2]) *Let c be a complex number with $|c| \le 1$, $c \ne -1$. If $f(z) = z + a_2z^2 + \dots$ is a regular function in U and*

$$\left| c|z|^2 + (1 - |z|^2)\frac{zf''(z)}{f'(z)} \right| \le 1,$$

for all $z \in U$, then the function f is regular and univalent in U.

Lemma B. [5] *Let α be a complex number,* $\operatorname{Re}\alpha > 0$, *and c a complex number, $|c| \le 1$, $c \ne -1$ and $f(z) = z + a_2z^2 + \dots$, a regular function in U.*

If

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \cdot \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1$$

for all $z \in U$*, then the function*

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}} = z + b_2 z^2 + \ldots$$

is regular and univalent in U*.*

Lemma C. [3] *If* $f \in A$ *satisfies the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1, \quad z \in U$$

then f *is univalent in* U*.*

Lemma D. [4] *Let* α *be a complex number,* $\mathrm{Re}\ \alpha > 0$*, and* $f \in A$*.*
If

$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in U$$

where β *complex number,* $\mathrm{Re}\ \beta \geq \mathrm{Re}\ \alpha$*,*
then the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{\frac{1}{\beta}}$$

is univalent in U*.*

2. Main results

Theorem 1. *Let* $M \geq 1$ *and* α *with* $\mathrm{Re}\ \alpha > 0$ *be a complex number,* $\alpha \neq 1$*, and* c *be a complex number, with* $|c| \leq 1$*,* $c \neq -1$*. Let the function* $g \in A$*, satisfies the conditions*

$$\left|\frac{g(z)}{z}\right| \leq 3M - 2 \tag{1}$$

$$\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| \leq \frac{1}{3M-2}, \tag{2}$$

for all $z \in U$*, and*

$$|c| + \frac{3|\alpha - 1|}{|\alpha|} \leq 1, \tag{3}$$

then the function

$$G_{\alpha,M}(z)=\left[\frac{\alpha}{M}\int_0^z u^{\frac{\alpha}{M}-1}\left[\frac{g(u)}{u}\right]^{\frac{\alpha-1}{M^2}}du\right]^{\frac{M}{\alpha}} \tag{4}$$

is in the class S.

Proof. We let

$$f(z)=\int_0^z\left[\frac{g(u)}{u}\right]^{\frac{\alpha-1}{M^2}}du,\quad z\in U. \tag{5}$$

The function is regular in U.

Differentiating (5), we obtain

$$f'(z)=\left[\frac{g(z)}{z}\right]^{\frac{\alpha-1}{M^2}},\quad z\in U,$$

$$f''(z)=\frac{\alpha-1}{M^2}\left[\frac{g(z)}{z}\right]^{\frac{\alpha-1}{M^2}-1}\cdot\frac{zg'(z)-g(z)}{z^2},\quad z\in U$$

and

$$\frac{zf''(z)}{f'(z)}=\frac{\alpha-1}{M^2}\left(\frac{zg'(z)}{g(z)}-1\right),\quad z\in U. \tag{6}$$

Using (1), (2), (3) and (6), we calculate

$$\left|c|z|^{2\frac{\alpha}{M}}+(1-|z|^{2\frac{\alpha}{M}})\frac{Mzf''(z)}{\alpha f'(z)}\right| \tag{7}$$

$$=\left|c\cdot|z|^{2\frac{\alpha}{M}}+(1-|z|^{2\frac{\alpha}{M}})\frac{\alpha-M}{\alpha}\left(\frac{zg'(z)}{g(z)}-1\right)\right|$$

$$\le|c|z|^{2\frac{\alpha}{M}}|+|1-|z|^{2\frac{\alpha}{M}}|\frac{|\alpha-1|}{|\alpha\cdot M|}\left|\frac{zg'(z)}{g(z)}-1\right|$$

$$\le|c|+\frac{|\alpha-1|}{|\alpha|M}\cdot\left|\frac{zg'(z)}{g(z)}-1\right|$$

$$\le|c|+\frac{|\alpha-1|}{|\alpha|M}\left[\left|\frac{zg'(z)}{g(z)}\right|+1\right]$$

$$\le|c|+\frac{|\alpha-1|}{|\alpha|M}\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|\cdot\left|\frac{g(z)}{z}\right|+1\right]$$

$$\leq |c| + \frac{|\alpha-1|}{|\alpha|M}\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|(3M-2)+1\right]$$

$$\leq |c| + \frac{|\alpha-1|}{|\alpha|M}\left[\left|\frac{z^2g'(z)}{g^2(z)}-1\right|(3M-2)+3M-2+1\right]$$

$$\leq |c| + \frac{|\alpha-1|}{|\alpha|M}[1+3M-2+1] = |c| + \frac{3|\alpha-1|}{|\alpha|M}\cdot M =$$

$$= |c| + 3\frac{|\alpha-1|}{|\alpha|} \leq 1.$$

From (7), using Lemma B, we have $G_{\alpha,M}$ is in the class S.

Remark 1. For $M=1$, the result was obtained in [6].

Remark 2. For $M=1$, the condition (2) expresses a sufficient condition for univalence of function g and this result can be found in [3, Lemma C].

Theorem 2. *Let* $M \geq 1$ *and* α *with* $\mathrm{Re}\ \alpha > 0$ *be a complex number,* $\alpha \neq 1$*, and* β *be a complex number with* $\mathrm{Re}\ \beta > \mathrm{Re}\ \alpha$*. Let the function* g *satisfies the condition*

$$\left|\frac{zg'(z)}{g(z)}-1\right| < \frac{M}{3} \tag{8}$$

for all $z \in U$*, and*

$$|\alpha| < 3\mathrm{Re}\ \alpha, \tag{9}$$

then the function

$$F_{\alpha,\beta,M}(z) = \left[\beta\int_0^z u^{\beta-1}\left[\frac{g(u)}{u}\right]^{\frac{\alpha}{M}}du\right]^{\frac{1}{\beta}} \tag{10}$$

is in the class S*.*

Proof. The function $F_{\alpha,\beta,M}$ given by (10) is regular in U.
We let

$$F_{\alpha,M}(z) = \int_0^z \left[\frac{g(u)}{u}\right]^{\frac{\alpha}{M}}, \quad z \in U. \tag{11}$$

Differentiating (11) we have

$$F'_{\alpha,M}(z) = \left[\frac{g(z)}{z}\right]^{\frac{\alpha}{M}}, \quad z \in U,$$

$$F''_{\alpha,M}(z) = \frac{\alpha}{M}\left[\frac{g(z)}{z}\right]^{\frac{\alpha}{M}-1} \cdot \frac{zg'(z) - g(z)}{z^2}, \quad z \in U$$

and

$$\frac{zF''_{\alpha,M}(z)}{F'_{\alpha,M}(z)} = \frac{\alpha}{M} \cdot \left[\frac{zg'(z)}{g(z)} - 1\right]. \tag{12}$$

Using (8), (9), (10), (12), we calculate

$$\frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha}\left|\frac{zF''_{\alpha,M}(z)}{F'_{\alpha,M}(z)}\right| \le$$

$$\le \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot \left|\frac{\alpha}{M} \cdot \left[\frac{zg'(z)}{g(z)} - 1\right]\right| =$$

$$= \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot \frac{|\alpha|}{M} \cdot \left|\frac{zg'(z)}{g(z)} - 1\right| \le$$

$$\le \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot \frac{|\alpha|}{M} \cdot \frac{M}{3} =$$

$$= \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot \frac{|\alpha|}{3} \le \frac{|\alpha|}{3\text{Re }\alpha} < 1.$$

From (12), using Lemma D we have that $F_{\alpha,\beta,M}$ is in the class S.

Remark 3. For $M = 1$, $\beta = 1$ the result was obtained in [2].

Theorem 3. *Let $M \ge 1$ and α with* Re $\alpha > 0$ *be a complex number, $\alpha \ne 1$, and β be a complex number with* Re $\beta >$ Re α. *Let the function g satisfies the condition*

$$|zg'(z)| \le M^2 \tag{13}$$

for all $z \in U$, and

$$\frac{|\alpha - 1|}{\text{Re }\alpha} < 1, \tag{14}$$

then the function

$$H_{\alpha,\beta,M}(z) = \left[\frac{\beta-1}{M^2}\int_0^z u^{\frac{\beta-1}{M^2}-1}(e^{g(u)})^{\frac{\alpha-1}{M^2}}\,du\right]^{\frac{M^2}{\beta-1}}, \quad z \in U \tag{15}$$

is in the class S.

Proof. The function $H_{\alpha,\beta,M}$ given by (15) is regular in U.

Let us consider the function

$$f(z) = \int_0^z (e^{g(u)})^{\frac{\alpha-1}{M^2}} du, \quad z \in U \tag{16}$$

which is regular in U.

Differentiating (16), we obtain

$$f'(z) = e^{g(z)\frac{\alpha-1}{M^2}}$$

$$f''(z) = \frac{\alpha-1}{M^2} g'(z) e^{g(z)\frac{\alpha-1}{M^2}}, \quad z \in U$$

and

$$\frac{zf''(z)}{f'(z)} = \frac{\alpha-1}{M^2} zg'(z), \quad z \in U. \tag{17}$$

Using (13), (14), (16) and (17), we calculate

$$\frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \left| \frac{zf''(z)}{f'(z)} \right| = \tag{18}$$

$$= \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot \left| \frac{\alpha-1}{M^2} zg'(z) \right| \le$$

$$\le \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot \frac{|\alpha-1|}{M^2} \cdot \left| zg'(z) \right| \le$$

$$\le \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot \frac{|\alpha-1|}{M^2} \cdot M^2 =$$

$$= \frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha} \cdot |\alpha-1| \le \frac{|\alpha-1|}{\text{Re }\alpha} < 1.$$

From (18) using Lemma D, we have $H_{\alpha,\beta,M}$ is in the class S.

Remark 5. For $M = 1$, $\beta = 1$, the result was obtained in [6].

Example 1. For $M = 5$, $\alpha = \frac{1}{2} + \frac{1}{3}i$, $\beta = \frac{10}{9} + \frac{1}{6}i$,
and $g \in \mathcal{A}$, $g(z) = z + \frac{1}{4}z^2$, $z \in U$, we have:
$\text{Re }\alpha > 0$, $\text{Re }\beta > \text{Re }\alpha$, $|\alpha| = \frac{13}{36} < \frac{1}{2} = 3\text{Re }\alpha$, and

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| = \left| \frac{z + \frac{1}{2}z^2}{z + \frac{1}{4}z^2} - 1 \right| =$$

$$= \left| \frac{-\frac{1}{2}z^2}{z + \frac{1}{4}z^2} \right| < \frac{\frac{1}{2}|z|}{|1 + \frac{1}{4}z|} <$$

$$< \frac{\frac{1}{2}}{\frac{9}{16}} < \frac{8}{9} < \frac{5}{3} = \frac{M}{3}.$$

Using Theorem 2, we obtain $F_{(\frac{1}{2}+\frac{1}{3}i),(\frac{10}{9}+\frac{1}{6}i),5}(z)$ is in the class S.

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