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INEXTENSIBLE FLOWS OF DEVELOPABLE SURFACES ASSOCIATED FOCAL CURVE OF HELICES IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

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Abstract. In this paper, we study inextensible flows of focal curves associated with developable surfaces in Euclidean 3-space $\mathbb{E}^3$. We give some characterizations for curvature and torsion of focal curves associated with developable surfaces in Euclidean 3-space $\mathbb{E}^3$. Finally, we show that if flow of this developable surface is inextensible then this surface is not minimal.

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1. Introduction

Curve design using splines is one of the most fundamental topics in CAGD. Inextensible flows of curves possess a beautiful shape preserving connection to their control polygon. They allow us the formulation of algorithms for processing, especially subdivision algorithms. Moreover, at least the curves of odd degree and maximal smoothness also arise as solutions of variational problems.

In the past two decades, for the need to explain certain physical phenomena and to solve practical problems, geometers and geometric analysis have begun to deal with curves and surfaces which are subject to various forces and which flow or evolve with time in response to those forces so that the metrics are changing. Now, various geometric flows have become one of the central topics in geometric analysis. Many authors have studied geometric flow problems.

On the other hand, ruled surfaces and especially developable surfaces are well-known and widely used in computer aided design and manufacture. Since these fields apply B-spline or NURBS surfaces as de facto standard description methods, it is highly desired to use these methods to construct ruled and developable surfaces from any type of data. These data can be scattered points or given lines or a set of tangent planes, as well. Since these special surfaces possess a wide range of applications, e.g., from ship hulls to sheet metal forming processes, one can find several algorithms solving this problem.

It is well known that developable surfaces play an important role in design in several branches of industry, such as naval and textile. Even architectural structures have been designed using developable surfaces. In these industries surfaces are designed which mimic properties of the materials that are used in production, which are intended to be deformed from plane sheets of metal or cloth just by folding, cutting or rolling, but not stretching. This sort of industrial procedures are less expensive or do not alter the properties of the material an therefore developable surfaces are favoured.

In this paper, we study inextensible flows of focal curves associated with developable surfaces in Euclidean 3-space $\mathbb{E}^3$. We give some characterizations for curvature and torsion of focal curves associated with developable surfaces in Euclidean 3-space $\mathbb{E}^3$. Finally, we show that if flow of developable surface is inextensible then this surface is not minimal.

2. Preliminaries

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by $\|a\| = \sqrt{\langle a, a \rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the curve γ in the space $\mathbb{E}^3$. For an arbitrary curve γ with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= -\tau \mathbf{N}, \end{aligned}$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{B} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = 0. \end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}(s)\|$ and $\tau(s) = -\langle \mathbf{N}, \mathbf{B}' \rangle$.

Torsion of the curve γ is given by the aid of the mixed product

$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

3. Inextensible Flows of Developable Surfaces Associated with Focal Curve of Helix in the E^3

For a unit speed curve γ, the curve consisting of the centers of the osculating spheres of γ is called the parametrized focal curve of γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ, we can write

$$C_\gamma(s) = (\gamma + c_1\mathbf{T} + c_2\mathbf{N})(s), \tag{3.1}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa},\ c_2 = \frac{c_1'}{\tau},\ \kappa \neq 0,\ \tau \neq 0. \tag{3.2}$$

Lemma 3.1. *Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit speed helix and C_γ its focal curve on $\mathbb{E}^3$. Then,*

$$c_1 = \frac{1}{\kappa} = \text{constant and } c_2 = 0. \tag{3.3}$$

Proof. Using (3.1) and (3.2), we get (3.3).

On the other hand, a ruled surface in $\mathbb{E}^3$ is (locally) the map $\Omega_{(\gamma,\delta)} : I \times \mathbb{R} \rightarrow \mathbb{E}^3$ defined by

$$\Omega_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s),$$

where $\gamma : I \longrightarrow \mathbb{E}^3$, $\delta : I \longrightarrow \mathbb{E}^3\backslash\{0\}$ are smooth mappings and I is an open interval or the unit circle $\mathbb{S}^1$. We call the base curve and the director curve. The straight lines $u \rightarrow \gamma(s) + u\delta(s)$ are called rulings of $\Omega_{(\gamma,\delta)}$.

Definition 3.2. *A smooth surface $\Omega_{(\gamma,\delta)}$ is called a developable surface if its Gaussian curvature K vanishes everywhere on the surface.*

Definition 3.3. *Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit speed curve. We define the following developable surface*

$$\Omega_{(C_\gamma,\gamma')}(s,u) = C_\gamma(s) + u\gamma'(s), \tag{3.4}$$

where $C_\gamma(s)$ is focal curve.

Definition 3.4. ([8]) *A surface evolution $\Omega(s,u,t)$ and its flow $\dfrac{\partial\Omega}{\partial t}$ are said to be inextensible if its first fundamental form $\{E,F,G\}$ satisfies*

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0. \tag{3.5}$$

This definition states that the surface $\Omega(s,u,t)$ is, for all time t, the isometric image of the original surface $\Omega(s,u,t_0)$ defined at some initial time t_0. For a developable surface, $\Omega(s,u,t)$ can be physically pictured as the parametrization of a waving flag. For a given surface that is rigid, there exists no nontrivial inextensible evolution.

Definition 3.5. *We can define the following one-parameter family of developable ruled surface*

$$\Omega(s,u,t) = C_\gamma(s,t) + u\gamma'(s,t). \tag{3.6}$$

Theorem 3.6. *Let Ω is the developable surface associated with focal curve in $\mathbb{E}^3$. $\dfrac{\partial\Omega}{\partial t}$ is inextensible, then*

$$\frac{\partial}{\partial t}\left(u^2\kappa^2 + c_1^2\tau^2\right) = 0. \tag{3.7}$$

Proof. Assume that $\Omega(s,u,t)$ be a one-parameter family of developable surface. We show that Ω is inextensible.

$$\begin{aligned} \Omega_s &= c_1\tau\mathbf{B} + u\kappa\mathbf{N}, \\ \Omega_u &= \mathbf{T}. \end{aligned}$$

If we compute first fundamental form $\{E,F,G\}$, we have

$$\begin{aligned} E &= \langle\Omega_s,\Omega_s\rangle = u^2\kappa^2 + c_1^2\tau^2, \\ F &= 0, \\ G &= 1. \end{aligned}$$

Using above system, we have

$$\begin{aligned}\frac{\partial E}{\partial t} &= 0,\\ \frac{\partial F}{\partial t} &= 0,\\ \frac{\partial G}{\partial t} &= 0.\end{aligned}$$

If $\frac{\partial \Omega}{\partial t}$ is inextensible, then we have (3.7).

Theorem 3.7. *Let Ω is the developable surface associated with focal curve in E^3. If flow of this developable surface is inextensible then this surface is not minimal.*

Proof. Assume that $\Omega(s,u,t) = C_\gamma(s,t) + u\gamma'(s,t)$ be a one-parameter family of developable ruled surface. Components of second fundamental form of developable surface are

$$\begin{aligned}h_{11} &= \tau\sqrt{[\frac{\tau}{\kappa}]^2 + u^2\kappa^2},\\ h_{12} &= -\tau,\\ h_{22} &= 0.\end{aligned}$$

On the other hand, components of metric

$$\begin{aligned}g_{11} &= [\frac{\tau}{\kappa}]^2 + u^2\kappa^2,\\ g_{12} &= 0,\\ g_{22} &= 1.\end{aligned}$$

So, the mean curvature of one-parameter family of developable ruled surface $X(s,u,t) = C_\gamma(s,t) + u\gamma'(s,t)$ is

$$\begin{aligned}H &= g^{ij}h_{ij}\\ &= \frac{\tau}{\sqrt{[\frac{\tau}{\kappa}]^2 + u^2\kappa^2}}.\end{aligned}$$

Ω is a minimal ruled surface in $\mathbb{E}^3$ if and only if $\tau = 0$.

By the use of (3.2) and above equation the proof is complete.

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