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TIMELIKE BIHARMONIC CURVES ACCORDING TO FLAT METRIC IN LORENTZIAN HEISENBERG GROUP HEIS^3

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Abstract. In this paper, we study timelike biharmonic curves according to flat metric in the Lorentzian Heisenberg group Heis^3. We characterize timelike biharmonic curves in terms of their curvature and torsion.

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1. Introduction

In their original 1964 paper [5], Eells and Sampson proposed an infinite-dimensional Morse theory on the manifold of smooth maps between Riemannian manifolds. Though their main results concern the Dirichlet energy, they also suggested other functionals. The interest encountered by harmonic maps, and to a lesser extent by p-harmonic maps, has overshadowed the study of other possibilities, e.g. exponential harmonicity . While the examples mentioned so far are all of first-order functionals, one can investigate problems involving higher-order functionals. A prime example of these is the bienergy, not only for the role it plays in elasticity and hydrodynamics, but also because it can be seen as the next stage of investigation, should the theory of harmonic maps fail.

Harmonic maps $f:(M,g)\longrightarrow(N,h)$ between manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M e(f)\,v_g, \tag{1.1}$$

where v_g is the volume form on (M,g) and

$$e(f)(x):=\frac{1}{2}\|df(x)\|^2_{T^*M\otimes f^{-1}TN}$$

is the energy density of f at the point $x\in M$.

Critical points of the energy functional are called harmonic maps.

The first variational formula of the energy gives the following characterization of harmonic maps: the map f is harmonic if and only if its tension field $\tau(f)$ vanishes identically, where the tension field is given by

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [5], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M \|\tau(f)\|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [8,9], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df, \tau(f))\,df \\ &= 0, \end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study timelike biharmonic curves according to flat metric in the Lorentzian Heisenberg group Heis^3. We characterize timelike biharmonic curves in terms of their curvature and torsion.

2.The Lorentzian Heisenberg Group Heis^3

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = dx^2 + (xdy + dz)^2 - ((1 - x)\,dy - dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \frac{\partial}{\partial y} + (1 - x)\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 0, \quad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

where the (i, j)*-element in the table above equals* $\nabla_{e_i} e_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So we obtain that

$$R(\mathbf{e}_1, \mathbf{e}_3) = R(\mathbf{e}_1, \mathbf{e}_2) = R(\mathbf{e}_2, \mathbf{e}_3) = 0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

3. Timelike Biharmonic Curves According to Flat Metric in the Lorentzian Heisenberg Group Heis^3

An arbitrary curve $\gamma : I \longrightarrow Heis^3$ is spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Let $\gamma : I \longrightarrow Heis^3$ be a unit speed timelike curve and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa_1 \mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa_1 \mathbf{T} + \kappa_2 \mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\kappa_2 \mathbf{N}, \end{aligned} \tag{3.1}$$

where κ_1, κ_2 are curvature function and torsion function, respectively.

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned}\mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3.\end{aligned}$$

Theorem 3.1. *If $\gamma : I \longrightarrow Heis^3$ is a unit speed timelike biharmonic curve according to flat metric, then*

$$\begin{aligned}\kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 - \kappa_2^2 &= 0, \\ \kappa_2 &= \text{constant}.\end{aligned} \tag{3.2}$$

Proof. Using Equation (3.1), we have

$$\begin{aligned}\tau_2(\gamma) &= \nabla^3_{\mathbf{T}}\mathbf{T} - \kappa_1 R(\mathbf{T},\mathbf{N})\mathbf{T} \\ &= (3\kappa_1'\kappa_1)\mathbf{T} + (\kappa_1'' + \kappa_1^3 - \kappa_1\kappa_2^2)\mathbf{N} + (2\kappa_2\kappa_1' + \kappa_1\kappa_2')\mathbf{B} - \kappa_1 R(\mathbf{T},\mathbf{N})\mathbf{T}.\end{aligned}$$

On the other hand, from Equation (2.4) we get

$$(3\kappa_1'\kappa_1)\mathbf{T} + (\kappa_1'' + \kappa_1^3 - \kappa_1\kappa_2^2)\mathbf{N} + (2\kappa_2\kappa_1' + \kappa_1\kappa_2')\mathbf{B} = 0. \tag{3.3}$$

Since $\kappa_1 \neq 0$ by the assumption that is non-geodesic

$$\begin{aligned}\kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 - \kappa_2^2 &= 0, \\ \kappa_2' &= 0.\end{aligned} \tag{3.4}$$

This completes the proof.

Corollary 3.2. *If $\gamma : I \longrightarrow Heis^3$ is a unit speed timelike biharmonic curve, then γ is a helix.*

Theorem 3.3. *Let $\gamma : I \longrightarrow Heis^3$ is a unit speed timelike biharmonic curve* *according to flat metric. Then the parametric equations of* γ *are*

$$\begin{aligned} x(s) &= \sinh\varphi s+\ell_1, \\ y(s) &= \frac{1}{\kappa_1}\cosh^2\varphi[\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]]+\ell_2, \\ z(s) &= -\frac{(-1+\ell_1+\sinh\varphi s)\cosh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\ &\quad +\frac{\cosh^2\varphi\sinh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]] \\ &\quad -\frac{\cosh\varphi(\sinh\varphi s+\ell_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]]+\ell_3, \end{aligned} \tag{3.5}$$

where $\ell,\ell_1,\ell_2,\ell_3$ *are constants of integration.*

Proof. Since γ is timelike biharmonic, γ is a timelike helix. So, without loss of generality, we take the axis of γ is parallel to the spacelike vector $\mathbf{e}_1$. Then,

$$g(\mathbf{T},\mathbf{e}_1)=T_1=\sinh\varphi, \tag{3.6}$$

where φ is constant angle.

The tangent vector can be written in the following form

$$\mathbf{T}=T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3. \tag{3.7}$$

On the other hand, the tangent vector $\mathbf{T}$ is a unit timelike vector, we get

$$\begin{aligned} T_2 &= \cosh\varphi\sinh\Omega, \\ T_3 &= \cosh\varphi\cosh\Omega, \end{aligned} \tag{3.8}$$

where Ω is an arbitrary function of s.

So, substituting the components T_1, T_2 and T_3 in the Equation (3.18), we have the following equation

$$\mathbf{T}=\sinh\varphi\mathbf{e}_1+\cosh\varphi\sinh\Omega\mathbf{e}_2+\cosh\varphi\cosh\Omega\mathbf{e}_3. \tag{3.9}$$

Using above equation and Frenet equations, we obtain

$$\Omega=\frac{\kappa_1 s}{\cosh\varphi}+\ell, \tag{3.10}$$

where ℓ is a constant of integration.

Thus Equation (3.9) and Equation (3.10), imply

$$\begin{aligned}\mathbf{T} &= \sinh\varphi\mathbf{e}_1+\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\mathbf{e}_2 \\ &\quad +\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\mathbf{e}_3.\end{aligned} \tag{3.11}$$

Using Equation (2.1) in Equation (3.11), we obtain

$$\begin{aligned}\mathbf{T} &= (\sinh\varphi,\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell], \\ &\quad (1-x)\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-x\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]).\end{aligned} \tag{3.12}$$

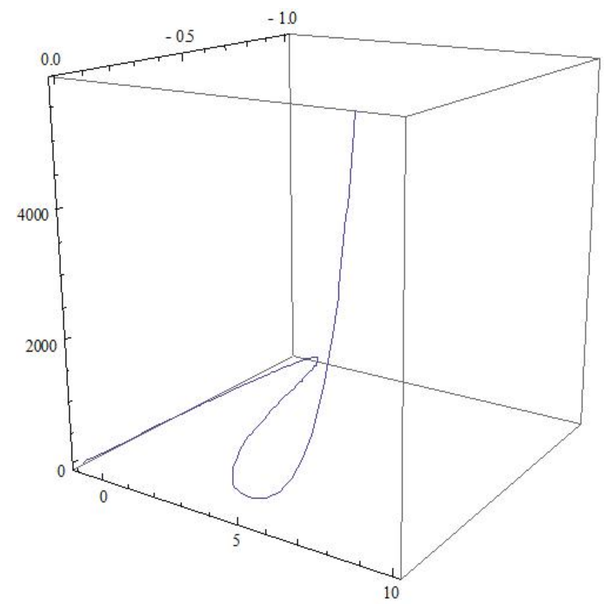
Also, from above Equation (3.11), we get

$$\begin{aligned}\mathbf{T} &= (\sinh\varphi,\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell], \\ &\quad (1-(\sinh\varphi s+\ell_1))\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-(\sinh\varphi s+\ell_1)\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]).\end{aligned} \tag{3.13}$$

Now Equation (3.13) becomes

$$\begin{aligned}\frac{dx}{ds} &= \sinh\varphi, \\ \frac{dy}{ds} &= \cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell], \\ \frac{dz}{ds} &= (1-(\sinh\varphi s+\ell_1))\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\ &\quad -(\sinh\varphi s+\ell_1)\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell].\end{aligned}$$

If we take integrate above system we have Equation (3.5). The proof is completed.



References

[1] L. Bianchi, *Ricerche sulle superficie elicoidali e sulle superficie a curvatura costante*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (1) 2 (1879), 285–341.

[2] R. Caddeo, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. 12 (2001), 867–876.

[3] R. Caddeo, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds in spheres,* Israel J. Math. 130 (2002), 109–123.

[4] R. Caddeo, S. Montaldo, P. Piu, *Biharmonic curves on a surface*, Rend. Mat. Appl. 21 (2001), 143–157.

[5] J. Eells, J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109–160.

[6] J. Happel, H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media,* Prentice-Hall, New Jersey, (1965).

[7] J. Inoguchi, *Submanifolds with harmonic mean curvature in contact 3-manifolds,* Colloq. Math. 100 (2004), 163–179.

[8] G.Y. Jiang, *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A 7 (1986), 130–144.

[9] G.Y. Jiang, *2-harmonic maps and their first and second variation formulas*, Chinese Ann. Math. Ser. A 7 (1986), 389–402.

[10] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[11] C. Oniciuc, *On the second variation formula for biharmonic maps to a sphere*, Publ. Math. Debrecen 61 (2002), 613–622.

[12] Y. L. Ou, *p-Harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps* , J. Geom. Phys. 56 (2006), 358-374.

[13] S. Rahmani, *Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois*, Journal of Geometry and Physics 9 (1992), 295-302.

[14] T. Sasahara, *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors,* Publ. Math. Debrecen 67 (2005), 285–303.

[15] T. Sasahara, *Stability of biharmonic Legendre submanifolds in Sasakian space forms*, preprint.

[16 E. Turhan, *Completeness of Lorentz Metric on 3-Dimensional Heisenberg Group*, International Mathematical Forum, 3, no. 13 (2008), 639 - 644.

[17] E. Turhan, T. Körpınar, *Characterize on the Heisenberg Group with left invariant Lorentzian metric*, Demonstratio Mathematica, 42 (2) (2009), 423-428

[18] E. Turhan, T. Körpınar, *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis*^{3}*,* Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

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