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SOME SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $U = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. In this paper, we introduce and study certain subclasses of analytic functions. Several inclusion relations of various subclasses of analytic functions, coefficient estimates , growth and distortion theorems, Hadamard Product, extreme points, integral means and inclusion properties are given. Moreover, some sufficient conditions for subordination and superordination of analytic functions, and application of $\Phi-$like function are also discussed.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the open unit disk $U = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$.

For a function $f \in \mathcal{A}$, $\alpha, \beta, \mu, \lambda \geq 0$ and $n \in N_o$ we define the differential operator, as follow:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1_\lambda(\alpha,\beta,\mu) f(z) &= (\frac{\alpha-\mu+\beta-\lambda}{\alpha+\beta}) f(z) + (\frac{\mu+\lambda}{\alpha+\beta}) z f'(z), \\ D^2_\lambda(\alpha,\beta,\mu) f(z) &= D(D^1_\lambda(\alpha,\beta,\mu) f(z)), \\ &\vdots \\ D^n_\lambda(\alpha,\beta,\mu) f(z) &= D(D^{n-1}_\lambda(\alpha,\beta,\mu) f(z)). \end{aligned} \tag{2}$$

If f is given by (1), then from (2), we see that

$$D_\lambda^n(\alpha,\beta,\mu)f(z) = z+\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n a_k z^k \qquad (3)$$

which generalizes many operators. Indeed, if in the definition of $D_\lambda^n(\alpha,\beta,\mu)f$ we substitute the following:

- $\beta=1$, $\mu=0$, we get
$D_\lambda^n(\alpha,1,0)f(z)=D^nf(z)=z+\sum_{k=2}^{\infty}(\frac{\alpha+\lambda(k-1)+1}{\alpha+1})^n a_k z^k$ of Aouf, El-Ashwah and El-Deeb differential operator[1].

- $\alpha=1$, $\beta=o$ and $\mu=0$, we get
$D_\lambda^n(1,0,0)f(z)=D^nf(z)=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))^n a_k z^k$ of Al-Oboudi differential operator[9].

- $\alpha=1$, $\beta=o$, $\mu=0$ and $\lambda=1$, we get
$D_1^n(1,0,0)f(z)=D^nf(z)=z+\sum_{k=2}^{\infty}(k)^n a_k z^k$ of Sălăgean's differential operator [5].

- $\alpha=1$, $\beta=1$, $\lambda=1$ and $\mu=0$, we get
$D_1^n(1,1,0)f(z) = D^nf(z) = z+\sum_{k=2}^{\infty}(\frac{k+1}{2})^n a_k z^k$ of Uralegaddi and Somanatha differential operator [2].

- $\beta=1$, $\lambda=1$ and $\mu=0$, we get
$D_1^n(\alpha,1,0)f(z)=D^nf(z)=z+\sum_{k=2}^{\infty}(\frac{k+\alpha}{\alpha+1})^n a_k z^k$ of Cho and Srivastava differential operator [3,4].

Now we define some new subclasses of analytic functions by using extended multiplier transformations operator.

Definition 1. *Let $f\in\mathcal{A}$. Then $f\in\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ if and only if*

$$\Re\bigg(\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{D_\lambda^n(\alpha,\beta,\mu)f(z)}\bigg)>\delta,\quad 0\le\delta<1,\ z\in U.$$

Definition 2. *Let $f\in\mathcal{A}$. Then $f\in\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ if and only if*

$$\Re\bigg(\frac{(z(D_\lambda^n(\alpha,\beta,\mu)f(z))')'}{(D_\lambda^n(\alpha,\beta,\mu)f(z))'}\bigg)>\delta,\quad 0\le\delta<1,\ z\in U.$$

Definition 3. ($\Phi-$Like Function)[6,11] *Let Φ be an analytic function in a domain containing $f(U)$, $\Phi(0)=0$ and $\Phi'(0)>0$. The function $f \in A$ is called Φ-like if*

$$\Re\left(\frac{zf'(z)}{\Phi(f(z))}\right) > 0, \quad z \in U.$$

This concept was introduced by Brickman [6] and established that a function $f \in \mathcal{A}$ is univalent if and only if f is $\Phi-$Like for some Φ.

Definition 4.[10] *Let Φ be an analytic function in a domain containing $f(U)$, $\Phi(0)=0$ and $\Phi'(0)=1$, and $\Phi(\omega) \neq 0$ for $\omega \in f(U) \setminus \{0\}$. Let q be a fixed analytic function in U, $q(0)=1$. The function $f \in \mathcal{A}$ is called Φ-Like with respect to q if*

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z), \quad z \in U.$$

Definition 5.[7] *Denote by Q the set of all functions f that are analytic and injective on $\overline{U} - E(f)$ where*

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Definition 6. (Subordination Principal) [8] *For two functions f and g, analytic in U, we say that the function f is subordinate to g in U, and write $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function w, which is analytic in U with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=g(w(z))$.*

Lemma 1.*[8] Let q(z) be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$,*

$$Q(z) = zq(z)\phi(q'(z)) \quad h(z) = \theta(q(z)) + Q(z),$$

suppose that $Q(z)$ is starlike univalent in $\boldsymbol{U}$ and $\Re(\frac{zh'(z)}{Q(z)}) > 0$ for $z \in U$ if p is analytic in U with $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then

$$p(z) \prec q(z),$$

and q is the best dominant

2.Characterization properties

In this section, we study the coefficient estimates and intend to prove the distortion theorems via two different techniques for the functions contained in the classes $\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ and $\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$.

Theorem 1. *If an analytic function f belongs to $\mathcal{A}$ satisfies the following condition i.e*

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq 1-\delta, \quad 0\leq\delta<1. \tag{4}$$

then $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$.

Proof. Let $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, then from Definition 1, we have

$$\Re\Big(\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{D_\lambda^n(\alpha,\beta,\mu)f(z)}\Big) > \delta, \quad 0\leq\delta<1, \ \ z\in U.$$

We also know that

$$\Re(w)>\alpha \quad \text{if and only if} \quad |1-\alpha+w|>|1+\alpha-w|,$$

this implies that

$$|(1-\delta)D_\lambda^n(\alpha,\beta,\mu)f(z)+z(D_\lambda^n(\alpha,\beta,\mu)f(z))'|$$
$$>|(1+\delta)D_\lambda^n(\alpha,\beta,\mu)f(z)-z(D_\lambda^n(\alpha,\beta,\mu)f(z))'|$$

such that

$$|(2-\delta)z+\sum_{k=2}^{\infty}(1+k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n a_k z^k|-$$

$$|\delta z+\sum_{k=2}^{\infty}(1-k+\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n a_k z^k|>0\cdot$$

So after simplification, we have

$$\leq(2-2\delta)-\sum_{k=2}^{\infty}(2k-2\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k|$$

or

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k|\leq(1-\delta)$$

is our required result.
Finally the result is sharp with extremal function f given by

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\delta}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} z^k.$$

Taking suitable values of parameters and function f in the class $\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, Theorem 1 gives us the following immediate results.

Corollary 1. *Let the function f defined by (1) be in the class $\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$. Then we have*

$$|a_k| \leq \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} \quad k \geq 2.$$

Corollary 2. *Let the hypotheses of Theorem 1, be satisfied. Then for $\delta = \mu = \lambda = 0$, we have*

$$|a_k| \leq \frac{1}{k}, \quad k \geq 2.$$

Theorem 2. *If an analytic function f belongs to $\mathcal{A}$ satisfies the following condition i.e*

$$\sum_{k=2}^{\infty} k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n |a_k| \leq 1-\delta, \quad 0 \leq \delta < 1\cdot \tag{5}$$

$$(f \in A, \alpha, \beta, \mu, \lambda \geq 0, \alpha+\beta \neq 0, n \in N_o)$$

then $f \in \mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$.

Proof. The proof is similar to that of the proof of Theorem 2.

Similarly, for a function f to be in the class $\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, we deduced a result from Theorem 2, as follow:

Corollary 3. *Let the function f defined by (1) be in the class $\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$. Then we have*

$$|a_k| \leq \frac{(1-\delta)}{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} \quad k \geq 2.$$

3. Growth and distortion theorems

Theorem 3. *If an analytic function f belongs to $\mathcal{A}$ satisfies the following condition i.e*

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq 1-\delta, \quad 0 \leq \delta < 1, \tag{6}$$

then $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ and

$$|f(z)| \geq |z| - \frac{1-\delta}{2-\delta}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n|z|^2$$

and

$$|f(z)| \leq |z| + \frac{1-\delta}{2-\delta}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n|z|^2.$$

Proof. Let $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, this implies that

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq 1-\delta, \quad 0 \leq \delta < 1.$$

But

$$(2-\delta)\Big(\frac{\alpha+(\mu+\lambda)+\beta}{\alpha+\beta}\Big)^n\sum_{k=2}^{\infty}|a_k|$$

$$\leq \sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq 1-\delta, \quad 0 \leq \delta < 1,$$

implies that

$$(2-\delta)\Big(\frac{\alpha+(\mu+\lambda)+\beta}{\alpha+\beta}\Big)^n\sum_{k=2}^{\infty}|a_k| \leq 1-\delta, \quad 0 \leq \delta < 1$$

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{1-\delta}{(2-\delta)\Big(\frac{\alpha+(\mu+\lambda)+\beta}{\alpha+\beta}\Big)^n} \quad 0 \leq \delta < 1.$$

Since

$$|f(z)| = |z + \sum_{k=2}^{\infty}a_k z^k|$$

$$|f(z)| \leq |z| + \sum_{k=2}^{\infty} |a_k||z^2|.$$

$$|f(z)| \leq |z| + \frac{1-\delta}{2-\delta}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n |z^2|.$$

Similarly

$$|f(z)| = |z + \sum_{k=2}^{\infty} a_k z^k|$$

$$|f(z)| \geq |z| - \sum_{k=2}^{\infty} |a_k||z^2|.$$

$$|f(z)| \geq |z| - \frac{1-\delta}{2-\delta}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n |z^2|.$$

Theorem 4. *Let the hypotheses of Theorem 2, be satisfied,then*

$$|f(z)| \geq |z| - \frac{1-\delta}{2(2-\delta)}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n |z|^2,$$

and

$$|f(z)| \leq |z| + \frac{1-\delta}{2(2-\delta)}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n |z|^2.$$

Proof. Working on the same lines as in the proof of Theorem 3, one can easily prove this theorem as well.

Theorem 5. *If an analytic function f belongs to $\mathcal{A}$ satisfies the following condition i.e*

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n |a_k| \leq 1-\delta, \quad 0 \leq \delta < 1, \tag{7}$$

then $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ and

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \geq |z| - \frac{1-\delta}{2-\delta}|z|^2,$$

and

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \leq |z| + \frac{1-\delta}{2-\delta}|z|^2.$$

Proof. Let $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, this implies that

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq 1-\delta, \quad 0 \leq \delta < 1.$$

But

$$(2-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n\sum_{k=2}^{\infty}|a_k|$$

$$\leq \sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq 1-\delta$$

implies that

$$(2-\delta)\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq 1-\delta, \quad 0 \leq \delta < 1$$

$$\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k| \leq \frac{1-\delta}{(2-\delta)}.$$

Because

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| = |z+\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n a_k z^k|$$

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \leq |z|+\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k||z^k|,$$

this implies that

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \leq |z|+\frac{1-\delta}{(2-\delta)}|z^k|,$$

similarly

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| = |z+\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n a_k z^k|$$

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \geq |z|-\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n|a_k||z^k|,$$

this implies that

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \geq |z|-\frac{1-\delta}{(2-\delta)}|z^k|.$$

Theorem 6. *Let the hypotheses of Theorem 5, be satisfied,then*

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \geq |z| - \frac{1-\delta}{2(2-\delta)}|z|^2,$$

and

$$|D_\lambda^n(\alpha,\beta,\mu)f(z)| \leq |z| + \frac{1-\delta}{2(2-\delta)}|z|^2.$$

Proof. Use the same method of the proof of Theorem 5 to get the result.

Theorem 7. *Let the hypotheses of Theorem 3, be satisfied, then*

$$|f'(z)| \geq 1 - 2\frac{1-\delta}{2-\delta}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n|z|$$

and

$$|f'(z)| \leq 1 + 2\frac{1-\delta}{2-\delta}(\frac{\alpha+\beta}{\alpha+\mu+\lambda+\beta})^n|z|.$$

Proof. The proof is similar to that of the proof of Theorem 3.

4.Extreme Points

Theorem 8. *(a). If $f_1(z) = z$, and*

$$f_i(z) = z + \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} z^i \quad i = 2,3,4\cdots$$

Then $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, if and only if it can be expressed in the form $f(z) = \sum_{i=1}^{\infty}\lambda_i f_i(z)$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{\infty}\lambda_i = 1$.

(b). If $f_1(z) = z$,

$$f_i(z) = z + \frac{(1-\delta)}{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} z^i \quad i = 2,3,4\cdots$$

Then $f \in \mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, if and only if it can be expressed in the form $f(z) = \sum_{i=1}^{\infty}\lambda_i f_i(z)$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{\infty}\lambda_i = 1$.

Proof. (a) Let $f(z) = \sum_{i=1}^{\infty}\lambda_i f_i(z),\ i = 1,2,3,\cdots\ \lambda_i \geq 0$ with $\sum_{i=1}^{\infty}\lambda_i = 1\cdot$ This implies that

$$f(z) = \sum_{i=1}^{\infty}\lambda_i f_i(z),$$

or

$$\begin{aligned}
f(z) &= \lambda_1(z) + \sum_{i=2}^{\infty} \lambda_i \Big(z + \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} z^i\Big), \\
f(z) &= \lambda_1(z) + \sum_{i=2}^{\infty} \lambda_i(z) + \sum_{i=2}^{\infty} \lambda_i \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} z^i), \\
f(z) &= \sum_{i=1}^{\infty} \lambda_i(z) + \sum_{i=2}^{\infty} \lambda_i \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} z^i), \\
f(z) &= z + \sum_{i=2}^{\infty} \lambda_i \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} z^i, \qquad (8)
\end{aligned}$$

Since

$$\sum_{i=2}^{\infty} \lambda_i \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} (k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n$$

$$= \sum_{i=2}^{\infty} \lambda_i (1-\delta)$$

$$= (1-\lambda_1)(1-\delta) < (1-\delta)$$

The condition (6) for $f(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z)$, is satisfied$\cdot$ Thus $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$. Conversely, we suppose that $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, since

$$|a_k| \leq \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} \quad k \geq 2.$$

We put

$$\lambda_i = \frac{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n}{(1-\delta)} a_i \quad k \geq 2,$$

and

$$\lambda_1 = 1 - \sum_{i=2}^{\infty} \lambda_i,$$

then

$$f(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z).$$

The proof of the second part of the Theorem 8 is similar to 1st part.

5.Integral means inequalities

For any two functions f and g analytic in U, f is said to be subordinate to g in U, denoted by $f \prec g$ if there exists a Schwarz function w analytic in U satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)),\ z \in U$.

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. In 1925, Littlewood [12] proved the following Subordination Theorem.

Theorem 9. *If f and g are any two functions, analytic in U, with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$, $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\mu d\theta \le \int_0^{2\pi} |g(z)|^\mu d\theta.$$

Theorem 10. *(a). Let $f \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, and f_k be defined by*

$$f_k(z) = z + \frac{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n}{(1-\delta)} z^k \quad k \ge 2.$$

If there exists an analytic function $w(z)$ given by

$$[w(z)]^{k-1} = \frac{(1-\delta)}{(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} \sum_{k=2}^{\infty} a_k z^{k-1},$$

then for $z = re^{i\theta}$, and $(0 < r < 1)$,

$$\int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \le \int_0^{2\pi} |f_k(re^{i\theta})|^\mu d\theta.$$

(b). Let $f \in \mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, and f_k be defined by

$$f_k(z) = z + \frac{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n}{(1-\delta)} z^k \quad k \ge 2.$$

If there exists an analytic function $w(z)$ given by

$$[w(z)]^{k-1} = \frac{(1-\delta)}{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} \sum_{k=2}^{\infty} a_k z^{k-1},$$

then for $z = re^{i\theta}$, *and* $(0 < r < 1)$,

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta \leq \int_0^{2\pi} |f_k(re^{i\theta})|^{\mu} d\theta.$$

Proof. (b). We have to show that

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta \leq \int_0^{2\pi} |f_k(re^{i\theta})|^{\mu} d\theta,$$

or

$$\int_0^{2\pi} |z + \sum_{k=2}^{\infty} a_k z^k|^{\mu} d\theta \leq \int_0^{2\pi} |z + \frac{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n}{(1-\delta)} z^k|^{\mu} d\theta,$$

or

$$\int_0^{2\pi} |1 + \sum_{k=2}^{\infty} a_k z^{k-1}|^{\mu} d\theta \leq \int_0^{2\pi} |1 + \frac{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n}{(1-\delta)} z^{k-1}|^{\mu} d\theta,$$

By using Theorem 9, it is enough to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n}{(1-\delta)} z^{k-1},$$

Now by taking

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n}{(1-\delta)} (w(z))^{k-1},$$

After simplification we get

$$[w(z)]^k = \frac{(1-\delta)}{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} \sum_{k=2}^{\infty} a_k z^{k-1},$$

this implies that $w(0) = 0$, and

$$\left|[w(z)]^k\right| = \Big|\frac{(1-\delta)}{k(k-\delta)\left(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\right)^n} \sum_{k=2}^{\infty} a_k z^{k-1}\Big|,$$

or

$$\left|[w(z)]^k\right| = \frac{(1-\delta)}{k(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n} \sum_{k=2}^{\infty} |a_k|\left|z^{k-1}\right|,$$

By using (4) we get

$$|[w(z)]^{k-1}| \leq |z| < 1.$$

The proof of the 1st part of the Theorem 10 is similar to second part.

6.Hadamard Product

Let $f, g \in \mathcal{A}$, where $f(z)$ is given in (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, then the modified Hadamard product$f * g$ is defined by $(f * g) = z + \sum_{k=2}^{\infty} a_k b_k z^k$.

Theorem 11.*(a). If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ then prove that $(f * g)(z) \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$.*

(b). If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$ then prove that $(f * g)(z) \in \mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$.

Proof. (a). Since it is given that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, this implies that

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n |a_k| \leq 1-\delta, \quad 0 \leq \delta < 1.$$

Similarly $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$, implies that

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n |b_k| \leq 1-\delta, \quad 0 \leq \delta < 1.$$

Since

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n |a_k||b_k|$$

$$\leq \sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n |a_k| \leq 1-\delta, \quad 0 \leq \delta < 1.$$

Implies that

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^{n}|a_k||b_k|\leq 1-\delta,\quad 0\leq\delta<1.$$

Or$(f*g)(z)\in\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta)$. The proof of the second part of the Theorem 11 is similar to 1st part.

7.Subordination and inclusion properties

Here, we investigate the inclusion properties of the new classes defined above. We also find out some subordination results by using the $\Phi-$like function.

Theorem 12. *Let* $0\leq\alpha_1\leq\alpha_2<1$, $0\leq\beta_1\leq\beta_2<1$, $0\leq\mu_1\leq\mu_2<1$, $0\leq\lambda_1\leq\lambda_2<1$, *and* $0\leq\delta_1\leq\delta_2<1$, *then*

- $\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_2)\subseteq\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_1)$
- $\mathcal{M}_{\lambda_2,k}^{\alpha,\beta,\mu}(\delta)\subseteq\mathcal{M}_{\lambda_2,k}^{\alpha,\beta,\mu}(\delta)$
- $\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu_2}(\delta)\subseteq\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu_1}(\delta)$
- $\mathcal{M}_{\lambda,k}^{\alpha_1,\beta,\mu}(\delta)\subseteq\mathcal{M}_{\lambda,k}^{\alpha_2,\beta,\mu}(\delta)$
- $\mathcal{M}_{\lambda,k}^{\alpha,\beta_1,\mu}(\delta)\subseteq\mathcal{M}_{\lambda,k}^{\alpha,\beta_2,\mu}(\delta)$.

Proof. To prove $\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_2)\subseteq\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_1)$, since we have $0\leq\delta_1\leq\delta_2<1$, this implies that

$$\delta_1\leq\delta_2,\ or\ (1-\delta_2)\leq(1-\delta_1),$$

so let

$$f\in\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_2),$$

then from Theorem 1, we have

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^{n}|a_k|\leq 1-\delta_2,\quad 0\leq\delta_2<1\cdot$$

This implies that

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^{n}|a_k|\leq 1-\delta_2\leq 1-\delta_1,$$

and therefore

$$\sum_{k=2}^{\infty}(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^{n}|a_k| \leq 1-\delta_1,$$

implies that

$$f(z)\in\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_1),$$

and hence

$$\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_2)\subseteq\mathcal{M}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_1).$$

The proof of the remaining parts of the Theorem 12 is similar to 1st part.

Theorem 13. *Let* $0\leq\alpha_1\leq\alpha_2<1$, $0\leq\beta_1\leq\beta_2<1$, $0\leq\mu_1\leq\mu_2<1$, $0\leq\lambda_1\leq\lambda_2<1$, *and* $0\leq\delta_1\leq\delta_2<1$, *then*

- $\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_2)\subseteq\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu}(\delta_1)$
- $\mathcal{N}_{\lambda_2,k}^{\alpha,\beta,\mu}(\delta)\subseteq\mathcal{N}_{\lambda_2,k}^{\alpha,\beta,\mu}(\delta)$
- $\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu_2}(\delta)\subseteq\mathcal{N}_{\lambda,k}^{\alpha,\beta,\mu_1}(\delta)$
- $\mathcal{N}_{\lambda,k}^{\alpha_1,\beta,\mu}(\delta)\subseteq\mathcal{N}_{\lambda,k}^{\alpha_2,\beta,\mu}(\delta)$
- $\mathcal{N}_{\lambda,k}^{\alpha,\beta_1,\mu}(\delta)\subseteq\mathcal{N}_{\lambda,k}^{\alpha,\beta_2,\mu}(\delta)\cdot$

Proof. To prove $\mathcal{N}_{\lambda,k}^{\alpha_1,\beta,\mu}(\delta)\subseteq\mathcal{N}_{\lambda,k}^{\alpha_2,\beta,\mu}(\delta)\cdot$, we have

$$0\leq\alpha_1\leq\alpha_2<1\Rightarrow\alpha_1\leq\alpha_2,$$

implies

$$1/\alpha_2+\beta\leq 1/\alpha_1+\beta\Rightarrow 1+\frac{(\mu+\lambda)(k-1)}{\alpha_2+\beta}\leq 1+\frac{(\mu+\lambda)(k-1)}{\alpha_1+\beta},$$

therefore

$$\sum_{k=2}^{\infty}k(k-\delta)\Big(\frac{\alpha_2+(\mu+\lambda)(k-1)+\beta}{\alpha_2+\beta}\Big)^{n}|a_k|$$
$$\leq\sum_{k=2}^{\infty}k(k-\delta)\Big(\frac{\alpha_1+(\mu+\lambda)(k-1)+\beta}{\alpha_1+\beta}\Big)^{n}|a_k|,$$

so let

$$f(z)\in\mathcal{N}_{\lambda,k}^{\alpha_1,\beta,\mu}(\delta),$$

then from Theorem 2, we have

$$\sum_{k=2}^{\infty} k(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha_1+\beta}\Big)^n |a_k| \le 1-\delta,$$

implies that

$$\sum_{k=2}^{\infty} k(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha_2+\beta}\Big)^n |a_k|$$

$$\le \sum_{k=2}^{\infty} k(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha_1+\beta}\Big)^n |a_k| \le 1-\delta\cdot$$

This show that

$$\sum_{k=2}^{\infty} k(k-\delta)\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha_2+\beta}\Big)^n |a_k| \le 1-\delta,$$

and implies

$$f(z) \in \mathcal{N}_{\lambda,k}^{\alpha_2,\beta,\mu}(\delta) \ \Rightarrow \mathcal{N}_{\lambda,k}^{\alpha_1,\beta,\mu}(\delta) \subseteq \mathcal{N}_{\lambda,k}^{\alpha_2,\beta,\mu}(\delta)\cdot$$

The proof of the remaining parts of the Theorem 13 is similar to 4th part.

Theorem 14. *Let $q \neq 0$ be univalent in U such that $zq'(z)/q(z)$ is starlike univalent in U and*

$$\Re\Bigg(1+\frac{\alpha}{\gamma}q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\Bigg) > 0, \quad \alpha,\gamma \in C, \quad \gamma \neq 0. \tag{9}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$\alpha(\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{\Phi(D_\lambda^n(\alpha,\beta,\mu)f(z))})+\gamma(1+\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))''}{(D_\lambda^n(\alpha,\beta,\mu)f(z))'}-\frac{z\Phi'(D_\lambda^n(\alpha,\beta,\mu)f(z))}{\Phi(D_\lambda^n(\alpha,\beta,\mu)f(z))})$$

$$\prec \alpha q(z)+\gamma zq'(z)/q(z),$$

then

$$\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{\Phi(D_\lambda^n(\alpha,\beta,\mu)f(z))} \prec q(z),$$

and q is the best dominant.

Proof. Let

$$p(z)=\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{\Phi(D_\lambda^n(\alpha,\beta,\mu)f(z))},$$

then after computation, we have

$$zp'(z)/p(z) = 1 + \frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))''}{(D_\lambda^n(\alpha,\beta,\mu)f(z))'} - \frac{z\Phi'(D_\lambda^n(\alpha,\beta,\mu)f(z))}{\Phi(D_\lambda^n(\alpha,\beta,\mu)f(z))},$$

which yields the following subordination

$$\alpha p(z) + \gamma zp'(z)/p(z) \prec \alpha q(z) + \gamma zq'(z)/q(z), \quad \alpha, \gamma \in C.$$

By setting,

$$\theta(\omega) = \alpha\omega \quad \phi(\omega) = \gamma/\omega, \quad \gamma \neq 0,$$

it can be easily observed that $\theta(\omega)$ is analytic in C and $\phi(\omega)$ is analytic in $C - \{0\}$ and that $\phi(\omega) \neq 0$ when $\omega \in C - \{0\}\cdot$ Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \gamma zq'(z)/q(z),$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \gamma zq'(z)/q(z),$$

we find that $Q(z)$ is starlike univalent in U and that

$$\Re(\frac{zh'(z)}{Q(z)} = \Re(1 + \frac{\alpha}{\gamma}q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}) > 0.$$

So by Lemma 1, we have

$$\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{\Phi(D_\lambda^n(\alpha,\beta,\mu)f(z))} \prec q(z).$$

We deduce the following result by taking Φ an identity function i.e. $\Phi(\omega) = \omega$ in Theorem 14, as follows:

Corollary 4. *Let $q(z) \neq 0$ be univalent in $U\cdot$ If q satisfies (9) and*

$$\alpha(\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{(D_\lambda^n(\alpha,\beta,\mu)f(z))}) + \gamma(1 + \frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))''}{(D_\lambda^n(\alpha,\beta,\mu)f(z))'} - \frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{(D_\lambda^n(\alpha,\beta,\mu)f(z))})$$

$$\prec \alpha q(z) + \gamma zq'(z)/q(z),$$

then

$$\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{(D_\lambda^n(\alpha,\beta,\mu)f(z))} \prec q(z),$$

and q is the best dominant.

Similarly, by taking suitable values of a function q in Theorem 14, we deduce Corollary 5 and Corollary 6 as follows:

Corollary 5. *If $f \in \mathcal{A}$ and assume that (9) holds then*

$$1 + \frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))''}{(D_\lambda^n(\alpha,\beta,\mu)f(z))'} \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)}$$

implies

$$\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{(D_\lambda^n(\alpha,\beta,\mu)f(z))} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B \le A \le 1,$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Corollary 6. *If $f \in \mathcal{A}$ and assume that (9) holds then*

$$1 + \frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))''}{(D_\lambda^n(\alpha,\beta,\mu)f(z))'} \prec \frac{1+z}{1-z} + \frac{2z}{(1+z)(1-z)},$$

implies,

$$\frac{z(D_\lambda^n(\alpha,\beta,\mu)f(z))'}{(D_\lambda^n(\alpha,\beta,\mu)f(z))} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

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