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ON $|(N,p,q)(E,1)|_k$ SUMMABILITY OF ORTHOGONAL SERIES

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Abstract. In this paper we obtain some sufficient conditions on $|(N,p,q)(E,1)|_k$, $(1 \leq k \leq 2)$, summability of an orthogonal series. These conditions are expressed in terms of the coefficients of the orthogonal series. Several important results are also deduced as corollaries.

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1. Introduction

Let $\{p_n\}$ and $\{p_n\}$ be two sequences of constants, real or complex, such that

$$\begin{aligned} P_n &= p_0 + p_1 + p_2 + \cdots + p_n = \sum_{v=0}^{n} p_v, \\ Q_n &= q_0 + q_1 + q_2 + \cdots + q_n = \sum_{v=0}^{n} q_v, \\ R_n &= p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 = \sum_{v=0}^{n} p_v q_{n-v}. \end{aligned}$$

For two given sequences $\{p_n\}$ and $\{p_n\}$ the convolution $(p*q)_n$ is defined by

$$R_n := (p*q)_n := \sum_{v=0}^{n} p_{n-v} q_v.$$

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of its $n-$th partial sums $\{s_n\}$. We write

$$t_n^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v.$$

If $R_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

If $t_n^{p,q} \to s$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} a_n$ is summable to s by generalized Nörlund method [1] and is denoted by

$$s_n \to s(N,p,q).$$

The necessary and sufficient conditions for (N,p,q) method of summability to be regular are

$$\sum_{v=0}^{n} |p_{n-v}q_v| = O(|R_n|),$$

and $p_{n-v} = o(|R_n|)$, as $n \to \infty$, for every fixed $v \geq 0$, for which $q_v \neq 0$.

The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable (N,p,q) if the series

$$\sum_{n=1}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N,p,q|.$$

The $|N,p,q|$ method of summability was introduced by Tanaka [5].

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a,b). We assume that $f(x)$ belongs to $L^2(a,b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x), \tag{1}$$

where $c_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0,1,2,\dots)$.

We write

$$R_n^j := \sum_{v=j}^{n} p_{n-v}q_v, \quad R_n^{n+1} = 0, \; R_n^0 = R_n.$$

Regarding to the series (1) Okuyama [6] has proved the following two theorems.

Theorem 1.1. *If the series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n\varphi_n(x)$$

is summable $|N,p,q|$ *almost everywhere.*

Theorem 1.2. *Let* $\{\Omega(n)\}$ *be a positive sequence such that* $\{\Omega(n)/n\}$ *is a non-increasing sequence and the series* $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ *converges. Let* $\{p_n\}$ *and* $\{q_n\}$ *be non-negative. If the series* $\sum_{n=1}^{\infty}|c_n|^2\Omega(n)w(n)$ *converges, then the orthogonal series* $\sum_{n=0}^{\infty}c_n\varphi_n(x)\in|N,p,q|$ *almost everywhere, where* $w(n)$ *is defined by*

$$w(j) := j^{-1}\sum_{n=j}^{\infty} n^2\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2.$$

Further, we denote by

$$E_n^1 = \frac{1}{2^n}\sum_{v=0}^{n}\binom{n}{v}s_v$$

the Euler transform of the sequence $\{s_n\}$.

If $E_n^1\to s$, as $n\to\infty$, then the series $\sum_{n=0}^{\infty}a_n$ is said to be $(E,1)$ summable to s [2].

The composition of the $t_n^{p,q}$ mean with E_n^1 mean is defined by equality

$$\begin{aligned} t_n^{p,q;E} &= \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_vE_v^1 \\ &= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{2^v}\sum_{j=0}^{v}\binom{v}{j}s_j. \end{aligned}$$

If $t_n^{p,q;E}\to s$, as $n\to\infty$, then the series $\sum_{n=0}^{\infty}a_n$ is said to be $(N,p,q)(E,1)$ summable to s [3].

We introduce the concept of the absolute $(N,p,q)(E,1)$ summability of order k, $(k=1,2,\dots)$, with the following definition.

The infinite series $\sum_{n=0}^{\infty}a_n$ is said to be absolutely summable $|(N,p,q)(E,1)|_k$ if for $k\geq 1$ the series

$$\sum_{n=1}^{\infty} n^{k-1}|t_n^{p,q;E}-t_{n-1}^{p,q;E}|^k$$

converges, and we write

$$\sum_{n=0}^{\infty} a_n \in |(N,p,q)(E,1)|_k.$$

The main purpose of the present paper is to study the $|(N,p,q)(E,1)|_k$ summability of the orthogonal series (1) for $1 \leq k \leq 2$.

Throughout K denotes a positive constant that it may depends only on k, and be different in different relations.

The following lemma due to Beppo Levi (see, for example [4]) is often used in the theory of functions. It will need us to prove main results.

Lemma 1.3. *If $f_n(t) \in L(E)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_E f_n(t)dt < \infty, \tag{2}$$

then the series

$$\sum_{n=1}^{\infty} f_n(t)$$

converges almost everywhere on E to a function $f(t) \in L(E)$. Moreover, the series (2) is also convergent to f in the norm of $L(E)$.

2. Main Results

First we put

$$H_v^{\mu} := \frac{1}{2^v} \sum_{j=\mu}^{v} \binom{v}{j} \quad \text{and} \quad \overline{R}_v^{\mu} := H_v^{\mu} R_v^{\mu}.$$

We prove the following theorem.

Theorem 2.1. *If the series*

$$\sum_{n=1}^{\infty} \left\{ n^{k-1} \sum_{\mu=1}^{n} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 |c_{\mu}|^2 \right\}^{\frac{k}{2}}$$

converges for $1 \leq k \leq 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is summable $|(N,p,q)(E,1)|_k$ almost everywhere.

Proof. Let $1 < k < 2$. For the $t_n^{p,q;E}(x)$ transform of the partial sums $s_j = \sum_{\mu=0}^{j} c_\mu \varphi_\mu(x)$ of the orthogonal series $\sum_{\mu=0}^{\infty} c_\mu \varphi_\mu(x)$ we have that

$$\begin{aligned}
t_n^{p,q;E}(x) &= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{2^v}\sum_{j=0}^{v}\binom{v}{j}\sum_{\mu=0}^{j}c_\mu\varphi_\mu(x)\\
&= \frac{1}{R_n}\sum_{v=0}^{n}p_{n-v}q_v\sum_{\mu=0}^{v}c_\mu\varphi_\mu(x)\frac{1}{2^v}\sum_{j=\mu}^{v}\binom{v}{j}\\
&= \frac{1}{R_n}\sum_{v=0}^{n}p_{n-v}q_v\sum_{\mu=0}^{v}H_v^\mu c_\mu\varphi_\mu(x)\\
&= \frac{1}{R_n}\sum_{\mu=0}^{n}H_n^\mu c_\mu\varphi_\mu(x)\sum_{v=\mu}^{n}p_{n-v}q_v\\
&= \frac{}{R_n}\sum_{\mu=0}^{n}H_n^\mu R_n^\mu c_\mu\varphi_\mu(x)\\
&= \frac{1}{R_n}\sum_{\mu=0}^{n}\overline{R}_n^\mu c_\mu\varphi_\mu(x).
\end{aligned}$$

Since

$$\frac{\overline{R}_n^0}{R_n}-\frac{\overline{R}_{n-1}^0}{R_{n-1}}=\frac{\overline{H}_n^0R_n^0}{R_n}-\frac{\overline{H}_{n-1}^0R_{n-1}^0}{R_{n-1}}=\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}-\frac{1}{2^{n-1}}\sum_{j=0}^{n-1}\binom{n-1}{j}=0,$$

then

$$\begin{aligned}
\bar{\triangle}t_n^{p,q;E}(x) &:= t_n^{p,q;E}(x)-t_{n-1}^{p,q;E}(x)\\
&= \frac{1}{R_n}\sum_{\mu=0}^{n}\overline{R}_n^\mu c_\mu\varphi_\mu(x)-\frac{1}{R_{n-1}}\sum_{\mu=0}^{n-1}\overline{R}_{n-1}^\mu c_\mu\varphi_\mu(x)\\
&= \frac{1}{R_n}\sum_{\mu=0}^{n}\overline{R}_n^\mu c_\mu\varphi_\mu(x)-\frac{1}{R_{n-1}}\sum_{\mu=0}^{n}\overline{R}_{n-1}^\mu c_\mu\varphi_\mu(x)\\
&= \sum_{\mu=1}^{n}\left(\frac{\overline{R}_n^\mu}{R_n}-\frac{\overline{R}_{n-1}^\mu}{R_{n-1}}\right)c_\mu\varphi_\mu(x).
\end{aligned}$$

Using the orthogonality, and Hölder's inequality with $p = \frac{2}{k} > 1$ and q such that

$p+q=pq$, we obtain

$$\begin{aligned}\int_a^b |\bar{\triangle} t_n^{p,q;E}(x)|^k dx &\leq (b-a)^{1-\frac{k}{2}} \left(\int_a^b |t_n^{p,q;E}(x) - t_{n-1}^{p,q;E}(x)|^2 dx \right)^{\frac{k}{2}} \\ &= (b-a)^{1-\frac{k}{2}} \left[\sum_{\mu=1}^{n} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 |c_\mu|^2 \right]^{\frac{k}{2}}.\end{aligned}$$

Whence, the series

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\bar{\triangle} t_n^{p,q;E}(x)|^k dx \leq K \sum_{n=1}^{\infty} n^{k-1} \left[\sum_{\mu=1}^{n} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 |c_\mu|^2 \right]^{\frac{k}{2}} \quad (3)$$

converges since the last does by the assumption. From this fact and since the functions $|\bar{\triangle} t_n^{p,q;E}(x)|$ are non-negative, then by the Lemma 1.3 the series

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\triangle} t_n^{p,q;E}(x)|^k$$

converges almost everywhere. The theorem for $k=1,2$ can be proved in a same way. Namely, for $k=2$ we apply only the orthogonality, until for $k=1$ we apply Schwarz's inequality. This completes the proof of the theorem.

We can specialize the sequences $\{p_n\}$ and $\{q_n\}$ so that $|(N,p,q)(E,1)|_k$ summability method reduces to some particular methods of the absolute summability. Most important particular cases of the $|(N,p,q)(E,1)|_k$ summability method are:

1) If $q_n=1$ for all n, then $|(N,p,q)(E,1)|_k$ summability reduces to $|(N,p_n)(E,1)|_k$ summability;

2) If $p_n=1/(n+1)$ and $q_n=1$ for all n, then $|(N,p,q)(E,1)|_k$ summability reduces to $|(N,1/(n+1))(E,1)|_k$ summability;

3) If $p_n=1$ for all n, then $|(N,p,q)(E,1)|_k$ summability reduces to $|(\overline{N},q_n)(E,1)|_k$ summability;

4) If $p_n=\binom{n+\alpha-1}{\alpha-1}$, $\alpha>0$, and $q_n=1$ for all n, then $|(N,p,q)(E,1)|_k$ summability reduces to $|(C,\alpha)(E,1)|_k$ summability.

From theorem 2.1, for three of the above cases, we have the following corollaries (the fourth one can be discussed in a similar way).

Corollary 2.2. *If the series*

$$\sum_{n=1}^{\infty}\left\{n^{k-1}\sum_{\mu=1}^{n}\left(\frac{H_n^{\mu}P_{n-\mu}}{P_n}-\frac{H_{n-1}^{\mu}P_{n-1-\mu}}{P_{n-1}}\right)^2|c_\mu|^2\right\}^{\frac{k}{2}}$$

converges for $1\le k\le 2$, *then the orthogonal series*

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|(N,p_n)(E,1)|_k$ *almost everywhere.*

Corollary 2.3. *If the series*

$$\sum_{n=1}^{\infty}\left\{n^{k-1}\sum_{\mu=1}^{n}\left[H_n^{\mu}\left(1-\frac{\mu}{n+1}\right)-H_{n-1}^{\mu}\left(1-\frac{\mu}{n}\right)\right]^2|c_\mu|^2\right\}^{\frac{k}{2}}$$

converges for $1\le k\le 2$, *then the orthogonal series*

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|(N,1/(n+1))(E,1)|_k$ *almost everywhere.*

Corollary 2.4. *If the series*

$$\sum_{n=1}^{\infty}\left\{n^{k-1}\sum_{\mu=1}^{n}\left[H_n^{\mu}\left(1-\frac{Q_{\mu-1}}{Q_n}\right)-H_{n-1}^{\mu}\left(1-\frac{Q_{\mu-1}}{Q_{n-1}}\right)\right]^2|c_\mu|^2\right\}^{\frac{k}{2}}$$

converges for $1\le k\le 2$, *then the orthogonal series*

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|(\overline{N},q_n)(E,1)|_k$ *almost everywhere.*

Now we shall prove a general theorem concerning to $|(N,p,q)(E,1)|_k$ summability of an orthogonal series which involves a positive sequence with some additional conditions.

To do this first we put

$$\mathcal{Q}^{(k)}(\mu) := \frac{1}{\mu^{\frac{2}{k}-1}} \sum_{n=\mu}^{\infty} n^{\frac{2}{k}} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 \tag{4}$$

then the following theorem holds true.

Theorem 2.5. *Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |a_n|^2 \Omega^{\frac{2}{k}-1}(n) \mathcal{Q}^{(k)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} a_n \varphi_n(x) \in |(N,p,q)(E,1)|_k$ almost everywhere, where $\mathcal{Q}^{(k)}(n)$ is defined by (4).*

Proof. Applying Hölder's inequality to the inequality (3) we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\bar{\triangle} t_n^{p,q;E}(x)|^k dx \le \\
&\le K \sum_{n=1}^{\infty} n^{k-1} \left[\sum_{\mu=1}^{n} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 |c_\mu|^2 \right]^{\frac{k}{2}} \\
&= K \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{\frac{2-k}{2}}} \left[n\left(\Omega(n)\right)^{\frac{2}{k}-1} \sum_{\mu=1}^{n} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 |c_\mu|^2 \right]^{\frac{k}{2}} \\
&\le K \left(\sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))} \right)^{\frac{2-k}{2}} \left[\sum_{n=1}^{\infty} n\left(\Omega(n)\right)^{\frac{2}{k}-1} \sum_{\mu=1}^{n} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 |c_\mu|^2 \right]^{\frac{k}{2}} \\
&\le K \left\{ \sum_{\mu=1}^{\infty} |c_\mu|^2 \sum_{n=\mu}^{\infty} n\left(\Omega(n)\right)^{\frac{2}{k}-1} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 \right\}^{\frac{k}{2}} \\
&\le K \left\{ \sum_{\mu=1}^{\infty} |c_\mu|^2 \left(\frac{\Omega(\mu)}{\mu} \right)^{\frac{2}{k}-1} \sum_{n=\mu}^{\infty} n^{\frac{2}{k}} \left(\frac{\overline{R}_n^{\mu}}{R_n} - \frac{\overline{R}_{n-1}^{\mu}}{R_{n-1}} \right)^2 \right\}^{\frac{k}{2}} \\
&= K \left\{ \sum_{\mu=1}^{\infty} |c_\mu|^2 \Omega^{\frac{2}{k}-1}(\mu) \mathcal{Q}^{(k)}(\mu) \right\}^{\frac{k}{2}},
\end{aligned}$$

which is finite by assumption. Using again the Lemma 1.3 we obtain the proof of the theorem.

Remark 2.6. *It should be noted that from Theorem 2.5 one also can obtain the versions of the corollaries 2.2–2.4.*

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