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CERTAIN SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH GENERALIZED SALAGEAN OPERATOR

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Abstract. In this paper, we investigate necessary and sufficient coefficient conditions, distortion bounds, extreme points and convex combination of a new subclass of harmonic univalent functions defined by a generalization of modified Salagean operator.

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1.Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are members of A. We call h the analytic part and g the co-analytic part of f. In this case, f is sense-preserving if $|h'(z)| > |g'(z)|$ in U. See [4].

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$.

To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1}$$

Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

Let SH_j denote the class of all functions $f = h + \bar{g} \in SH$ such that h and g has the form

$$h(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=j}^{\infty} b_k z^k, \tag{2}$$

where $j \in \mathbb{N}$ is fixed. Clearly, $SH := SH_1$.

In 1984 Clunie and Sheil-Small [4], investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz [1], Silverman [9], Silverman and Silvia [10], Jahangiri [5] studied the harmonic univalent functions. More recently, Ahuja et al. [2] investigated the convolutions of special harmonic univalent functions and Ahuja [3] studied on connections between harmonic mappings and hypergeometric functions.

The differential operator D^n ($n \in \mathbb{N}_0$) was introduced by Salagean [8]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [6] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

For $f = h + \overline{g}$ given by (1), we defined [12] generalization of the modified Salagean operator of f :

$$D_\lambda^0 f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D_\lambda^1 f(z) = (1-\lambda) D^0 f(z) + \lambda D^1 f(z), \quad \lambda \geq 0, \tag{3}$$

$$D_\lambda^n f(z) = D_\lambda^1 \left(D_\lambda^{n-1} f(z) \right). \tag{4}$$

If f is given by (1) , then from (3) and (4) we see that

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [\lambda(k-1)+1]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} [\lambda(k+1)-1]^n \overline{b_k z^k}. \tag{5}$$

When $\lambda = 1$, we get modified Salagean differential operator [6]. If we take the co-analytic part of $f = h + \overline{g}$ of the form (1) is identically zero, $D_\lambda^n f$ reduces to the Al-Oboudi operator [7].

Denote by $SH_j(\lambda, m, n, \alpha)$ the subclass of SH consisting of functions f of the form (2) that satisfy the condition

$$\operatorname{Re}\left(\frac{D_\lambda^{m+n} f(z)}{D_\lambda^n f(z)} \right) \geq \alpha, \quad 0 \leq \alpha < 1, \; m \in \mathbb{N}, \; n \in \mathbb{N}_0, \tag{6}$$

where $D_\lambda^n f(z)$ is defined by (5) and

$$D_\lambda^{m+n} f(z) = z + \sum_{k=2}^{\infty} [\lambda(k-1)+1]^{m+n} a_k z^k + (-1)^{m+n} \sum_{k=1}^{\infty} [\lambda(k+1)-1]^{m+n} \overline{b_k z^k}.$$

We let the subclass $\overline{SH}_j(\lambda, m, n, \alpha)$ consisting of harmonic functions $f_{m,n} = h + \overline{g}_{m,n}$ in SH so that h and $g_{m,n}$ are of the form

$$h(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \; g_{m,n}(z) = (-1)^{m+n-1} \sum_{k=j}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{7}$$

By suitably specializing the parametres, the classes $SH_j(\lambda, m, n, \alpha)$ reduces to the various subclasses of harmonic univalent functions. Such as,
(i) $SH_1(1,1,0,0) = SH^*(0)$ ([1], [9], [10]),
(ii) $SH_1(1,1,0,\alpha) = SH^*(\alpha)$ ([5]),
(iii) $SH_1(1,1,1,0) = KH(0)$ ([1], [9], [10]),
(iv) $SH_1(1,1,1,\alpha) = KH(\alpha)$ ([5]),
(v) $SH_1(1,1,n,\alpha) = H(n,\alpha)$ ([6]),
(vi) $SH_1(1, m-n, n, \alpha) = S_H(m,n,\alpha)$ ([11]).

The object of the present paper is to investigate the various properties of harmonic univalent functions belonging to the subclasses $SH_j(\lambda, m, n, \alpha)$ and $\overline{SH}_j(\lambda, m, n, \alpha)$. We extend the results of [6] by generalizing the operator and we also extend the results of previous talk of authors [12]. Necessary and sufficient coefficient conditions, distortion bounds, extreme points and convex combination of above mentioned class are derived.

2.Main Results

In our first theorem, we introduce a sufficient coefficient condition for harmonic functions in $SH_j(\lambda, m, n, \alpha)$.

Theorem 1. *Let $f = h + \overline{g}$ be so that h and g are given by (2). Furthermore, let*

$$\sum_{k=j+1}^{\infty} \left([\lambda(k-1)+1]^{m+n} - \alpha \left[\lambda(k-1)+1\right]^{n} \right) |a_k|$$
$$+ \sum_{k=j}^{\infty} \left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha \left[\lambda(k+1)-1\right]^{n} \right) |b_k| \leq 1 - \alpha, \tag{8}$$

where $\lambda \geq 1$, $0 \leq \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$. Then f is sense-preserving, harmonic univalent in U and $f \in SH_j(\lambda, m, n, \alpha)$.

Proof. If $z_1 \neq z_2$,

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum\limits_{k=j}^{\infty} b_k\left(z_1^k-z_2^k\right)}{(z_1-z_2)+\sum\limits_{k=j+1}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right| \\
&> 1-\frac{\sum\limits_{k=j}^{\infty} k\left|b_k\right|}{1-\sum\limits_{k=j+1}^{\infty} k\left|a_k\right|} \\
&\geq 1-\frac{\sum\limits_{k=j}^{\infty} \frac{\left([\lambda(k+1)-1]^{m+n}-(-1)^m\alpha[\lambda(k+1)-1]^n\right)}{1-\alpha}\left|b_k\right|}{1-\sum\limits_{k=j+1}^{\infty} \frac{\left([\lambda(k-1)+1]^{m+n}-\alpha[\lambda(k-1)+1]^n\right)}{1-\alpha}\left|a_k\right|} \geq 0,
\end{aligned}$$

which proves univalence. Note that f is sense preserving in U. This is because

$$\begin{aligned}
\left|h'(z)\right| &\geq 1-\sum_{k=j+1}^{\infty} k\left|a_k\right|\left|z\right|^{k-1} \\
&> 1-\sum_{k=j+1}^{\infty} \frac{\left([\lambda(k-1)+1]^{m+n}-\alpha\left[\lambda(k-1)+1\right]^n\right)}{1-\alpha}\left|a_k\right| \\
&\geq \sum_{k=j}^{\infty} \frac{\left([\lambda(k+1)-1]^{m+n}-(-1)^m\alpha\left[\lambda(k+1)-1\right]^n\right)}{1-\alpha}\left|b_k\right| \\
&> \sum_{k=j}^{\infty} k\left|b_k\right|\left|z\right|^{k-1} \geq \left|g'(z)\right|.
\end{aligned}$$

Using the fact that $\operatorname{Re} w \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|(1-\alpha)D_\lambda^n f(z)+D_\lambda^{m+n} f(z)\right|-\left|(1+\alpha)D_\lambda^n f(z)-D_\lambda^{m+n} f(z)\right| \geq 0. \qquad (9)$$

Substituting for $D_\lambda^{m+n} f(z)$and $D_\lambda^n f(z)$ in (9), we obtain

$$\begin{aligned}
&\left|(1-\alpha)D_\lambda^n f(z) + D_\lambda^{m+n} f(z)\right| - \left|(1+\alpha)D_\lambda^n f(z) - D_\lambda^{m+n} f(z)\right| \\
\geq\ & 2(1-\alpha)\left|z\right| - \sum_{k=j+1}^{\infty}\left([\lambda(k-1)+1]^n(1-\alpha) + [\lambda(k-1)+1]^{m+n}\right)|a_k|\,|z|^k \\
&-\sum_{k=j}^{\infty}\left|[\lambda(k+1)-1]^n(1-\alpha) + (-1)^m[\lambda(k+1)-1]^{m+n}\right||b_k|\,|z|^k \\
&-\sum_{k=j+1}^{\infty}\left([\lambda(k-1)+1]^{m+n} - (1+\alpha)[\lambda(k-1)+1]^n\right)|a_k|\,|z|^k \\
&-\sum_{k=j}^{\infty}\left|(-1)^m[\lambda(k+1)-1]^{m+n} - (1+\alpha)[\lambda(k+1)-1]^n\right||b_k|\,|z|^k
\end{aligned}$$

$$= \begin{cases}
2(1-\alpha)|z| - 2\sum\limits_{k=j+1}^{\infty}\left([\lambda(k-1)+1]^{m+n} - \alpha[\lambda(k-1)+1]^n\right)|a_k|\,|z|^k \\
\quad -2\sum\limits_{k=j}^{\infty}\left([\lambda(k+1)-1]^{m+n} + \alpha[\lambda(k+1)-1]^n\right)|b_k|\,|z|^k, \quad m \text{ is odd} \\
2(1-\alpha)|z| - 2\sum\limits_{k=j+1}^{\infty}\left([\lambda(k-1)+1]^{m+n} - \alpha[\lambda(k-1)+1]^n\right)|a_k|\,|z|^k \\
\quad -2\sum\limits_{k=j}^{\infty}\left([\lambda(k+1)-1]^{m+n} - \alpha[\lambda(k+1)-1]^n\right)|b_k|\,|z|^k, \quad m \text{ is even}
\end{cases}$$

$$\begin{aligned}
\geq\ & 2(1-\alpha)|z|\left\{1 - \sum_{k=j+1}^{\infty}\frac{\left([\lambda(k-1)+1]^{m+n} - \alpha[\lambda(k-1)+1]^n\right)}{1-\alpha}|a_k|\,|z|^{k-1}\right. \\
&\left.-\sum_{k=j}^{\infty}\frac{\left([\lambda(k+1)-1]^{m+n} - (-1)^m\alpha[\lambda(k+1)-1]^n\right)}{1-\alpha}|b_k|\,|z|^{k-1}\right\} \\
>\ & 2(1-\alpha)\left\{1 - \sum_{k=j+1}^{\infty}\frac{\left([\lambda(k-1)+1]^{m+n} - \alpha[\lambda(k-1)+1]^n\right)}{1-\alpha}|a_k|\right. \\
&\left.-\sum_{k=j}^{\infty}\frac{\left([\lambda(k+1)-1]^{m+n} - (-1)^m\alpha[\lambda(k+1)-1]^n\right)}{1-\alpha}|b_k|\right\}.
\end{aligned}$$

This last expression is non-negative by (8), and so the proof is complete.

Theorem 2. *Let* $f_{m,n} = h + \overline{g}_{m,n}$ *be given by (7). Then* $f_{m,n} \in \overline{SH}_j(\lambda, m, n, \alpha)$ *if and only if*

$$\sum_{k=j+1}^{\infty} \left([\lambda(k-1)+1]^{m+n} - \alpha [\lambda(k-1)+1]^{n} \right) a_k \tag{10}$$

$$+\sum_{k=j}^{\infty} \left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha [\lambda(k+1)-1]^{n} \right) b_k \leq 1-\alpha,$$

where $\lambda \geq 1$, $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $0 \leq \alpha < 1$.

Proof. The "if" part follows from Theorem 1 upon noting that $\overline{SH}_j(\lambda, m, n, \alpha) \subset SH_j(\lambda, m, n, \alpha)$. For the "only if" part, we show that $f_{m,n} \notin \overline{SH}_j(\lambda, m, n, \alpha)$ if the condition (10) does not hold.

Note that a necessary and sufficient condition for $f_{m,n} = h + \overline{g}_{m,n}$ given by (7), to be in $\overline{SH}_j(\lambda, m, n, \alpha)$ is that the condition (6) to be satisfied. This is equivalent to

$$\operatorname{Re}\left\{ \frac{(1-\alpha)z - \sum\limits_{k=j+1}^{\infty} \left([\lambda(k-1)+1]^{m+n} - \alpha [\lambda(k-1)+1]^{n} \right) a_k z^k}{z - \sum\limits_{k=j+1}^{\infty} [\lambda(k-1)+1]^{n} a_k z^k + (-1)^{m+2n-1} \sum\limits_{k=j}^{\infty} [\lambda(k+1)-1]^{n} b_k \overline{z}^k} \right.$$

$$\left. \frac{-\sum\limits_{k=j}^{\infty} \left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha [\lambda(k+1)-1]^{n} \right) b_k \overline{z}^k}{z - \sum\limits_{k=j+1}^{\infty} [\lambda(k-1)+1]^{n} a_k z^k + (-1)^{m+2n-1} \sum\limits_{k=j}^{\infty} [\lambda(k+1)-1]^{n} b_k \overline{z}^k} \right\} \geq 0. \tag{11}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\alpha) - \sum\limits_{k=j+1}^{\infty} \left([\lambda(k-1)+1]^{m+n} - \alpha [\lambda(k-1)+1]^{n} \right) a_k \, r^{k-1}}{1 - \sum\limits_{k=j+1}^{\infty} [\lambda(k-1)+1]^{n} a_k \, r^{k-1} - (-1)^m \sum\limits_{k=j}^{\infty} [\lambda(k+1)-1]^{n} b_k \, r^{k-1}}$$

$$\frac{-\sum\limits_{k=j}^{\infty} \left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha [\lambda(k+1)-1]^{n} \right) b_k \, r^{k-1}}{1 - \sum\limits_{k=j+1}^{\infty} [\lambda(k-1)+1]^{n} a_k \, r^{k-1} - (-1)^m \sum\limits_{k=j}^{\infty} [\lambda(k+1)-1]^{n} b_k \, r^{k-1}} \geq 0. \tag{12}$$

If the condition (10) does not hold, then the numerator in (12) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in

(12) is negative. This contradicts the required condition for $f_{m,n} \in \overline{SH}_j(\lambda, m, n, \alpha)$ and so the proof is complete.

Theorem 3. *Let $f_{m,n}$ be given by (7). Then $f_{m,n} \in \overline{SH}_j(\lambda, m, n, \alpha)$ if and only if*

$$f_{m,n}(z) = \sum_{k=j}^{\infty} \left(X_k h_k(z) + Y_k g_{m,n_k}(z) \right),$$

where $h_j(z) = z$,

$h_k(z) = z - \frac{1-\alpha}{\left([\lambda(k-1)+1]^{m+n} - \alpha[\lambda(k-1)+1]^n\right)} z^k \ (k = j+1, j+2, ...)$,

$g_{m,n_k}(z) = z + (-1)^{m+n-1} \frac{1-\alpha}{\left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha[\lambda(k+1)-1]^n\right)} \overline{z}^k$

$(k = j, j+1, j+2, ...)$,

$\sum_{k=j}^{\infty} (X_k + Y_k) = 1, \ X_k \geq 0, \ Y_k \geq 0.$

In particular, the extreme points of $\overline{SH}_j(\lambda, m, n, \alpha)$ are $\{h_k\}$ and $\{g_{m,n_k}\}$.

Proof. For functions $f_{m,n}$ of the form (7) we have

$$\begin{aligned} f_{m,n}(z) &= \sum_{k=j}^{\infty} \left(X_k h_k(z) + Y_k g_{m,n_k}(z) \right) \\ &= \sum_{k=j}^{\infty} (X_k + Y_k) z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{\left([\lambda(k-1)+1]^{m+n} - \alpha [\lambda(k-1)+1]^n\right)} X_k z^k \\ &\quad + (-1)^{m+n-1} \sum_{k=j}^{\infty} \frac{1-\alpha}{\left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha [\lambda(k+1)-1]^n\right)} Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=j+1}^{\infty} \frac{\left([\lambda(k-1)+1]^{m+n} - \alpha [\lambda(k-1)+1]^n\right)}{1-\alpha} \\ &\times \left(\frac{1-\alpha}{\left([\lambda(k-1)+1]^{m+n} - \alpha [\lambda(k-1)+1]^n\right)} X_k \right) \\ &+ \sum_{k=j}^{\infty} \frac{\left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha [\lambda(k+1)-1]^n\right)}{1-\alpha} \end{aligned}$$

$$\times\left(\frac{1-\alpha}{\left([\lambda(k+1)-1]^{m+n}-(-1)^{m}\alpha\left[\lambda(k+1)-1\right]^{n}\right)}Y_k\right)$$
$$=\sum_{k=j+1}^{\infty}X_k+\sum_{k=j}^{\infty}Y_k=1-X_j\leq 1,$$

and so $f_{m,n}\in\overline{SH}_j(\lambda,m,n,\alpha)$.

Conversely, if $f_{m,n}\in\overline{SH}_j(\lambda,m,n,\alpha)$, then

$$a_k\leq\frac{1-\alpha}{\left([\lambda(k-1)+1]^{m+n}-\alpha\left[\lambda(k-1)+1\right]^{n}\right)}$$

and

$$b_k\leq\frac{1-\alpha}{\left([\lambda(k+1)-1]^{m+n}-(-1)^{m}\alpha\left[\lambda(k+1)-1\right]^{n}\right)}.$$

Set

$$X_k=\frac{\left([\lambda(k-1)+1]^{m+n}-\alpha\left[\lambda(k-1)+1\right]^{n}\right)}{1-\alpha}a_k,\ (k=j+1,j+2,...)$$

$$Y_k=\frac{\left([\lambda(k+1)-1]^{m+n}-(-1)^{m}\alpha\left[\lambda(k+1)-1\right]^{n}\right)}{1-\alpha}b_k,\ (k=j,j+1,j+2,...)$$

and

$$X_j=1-\left(\sum_{k=j+1}^{\infty}X_k+\sum_{k=j}^{\infty}Y_k\right)$$

where $X_j\geq 0$. Then, as required, we obtain

$$f_{m,n}(z)=X_jz+\sum_{k=j+1}^{\infty}X_kh_k(z)+\sum_{k=j}^{\infty}Y_kg_k(z).$$

Theorem 4. *Let $f_{m,n}\in\overline{SH}_j(\lambda,m,n,\alpha)$. Then for $|z|=r<1$ we have* $|f_{m,n}(z)|\leq(1+b_j)\,r+\left(\frac{(1-\alpha)}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^{n}}-\frac{[\lambda(j+1)-1]^{m+n}-(-1)^{m}\alpha[\lambda(j+1)-1]^{n}}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^{n}}b_j\right)r^{j+1}$, *and*

$|f_n(z)|\geq(1-b_j)\,r-\left(\frac{(1-\alpha)}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^{n}}-\frac{[\lambda(j+1)-1]^{m+n}-(-1)^{m}\alpha[\lambda(j+1)-1]^{n}}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^{n}}b_j\right)r^{j+1}.$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_{m,n}\in\overline{SH}_j(\lambda,m,n,\alpha)$. Taking the absolute value of $f_{m,n}$ we have

$$\begin{aligned} |f_{m,n}(z)| &\leq (1+b_j)\,r + \sum_{k=j+1}^{\infty} (a_k+b_k)\,r^k \\ &\leq (1+b_j)\,r + \sum_{k=j+1}^{\infty} (a_k+b_k)\,r^{j+1} \\ &= (1+b_j)\,r + \frac{1-\alpha}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n} \\ &\quad \times \sum_{k=j+1}^{\infty} \frac{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n}{1-\alpha}(a_k+b_k)\,r^{j+1} \end{aligned}$$

$$\begin{aligned} &\leq (1+b_j)\,r + \frac{(1-\alpha)r^{j+1}}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n} \\ &\quad \times \sum_{k=j+1}^{\infty}\left(\frac{\left([\lambda(k-1)+1]^{m+n}-\alpha\,[\lambda(k-1)+1]^n\right)}{1-\alpha}a_k \right.\\ &\quad \left. + \frac{\left([\lambda(k+1)-1]^{m+n}-(-1)^m\alpha\,[\lambda(k+1)-1]^n\right)}{1-\alpha}b_k\right) \\ &\leq (1+b_j)\,r + \frac{(1-\alpha)r^{j+1}}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n} \\ &\quad \times \left(1-\frac{[\lambda(j+1)-1]^{m+n}-(-1)^m\alpha\,[\lambda(j+1)-1]^n}{1-\alpha}b_j\right) \\ &= (1+b_j)\,r \\ &\quad + \left(\frac{(1-\alpha)}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n}-\frac{[\lambda(j+1)-1]^{m+n}-(-1)^m\alpha[\lambda(j+1)-1]^n}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n}b_j\right)r^{j+1}. \end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary 1. *Let $f_{m,n}$ of the form (7) be so that $f_{m,n}\in\overline{SH}_j(\lambda,m,n,\alpha)$. Then*

$$\left\{w:|w|<\frac{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n+1-\alpha}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n} -\frac{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n-[\lambda(j+1)-1]^{m+n}+(-1)^n\alpha\,[\lambda(j+1)-1]^n}{(\lambda j+1)^{m+n}-\alpha(\lambda j+1)^n}b_j\right\} \subset f_{m,n}(U).$$

Theorem 5. *The class* $\overline{SH}_j(\lambda, m, n, \alpha)$ *is closed under convex combinations.*

Proof. for $i \geq j$, let $f_{m,n_i} \in \overline{SH}_j(\lambda, m, n, \alpha)$, where f_{m,n_i} is given by

$$f_{m,n_i}(z) = z - \sum_{k=j+1}^{\infty} a_{k_i} z^k + (-1)^{m+n-1} \sum_{k=j}^{\infty} b_{k_i} \overline{z}^k.$$

Then by (10),

$$\sum_{k=j+1}^{\infty} \frac{\left([\lambda(k-1)+1]^{m+n} - \alpha\left[\lambda(k-1)+1\right]^n\right)}{1-\alpha} a_{k_i} + \sum_{k=j}^{\infty} \frac{\left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha\left[\lambda(k+1)-1\right]^n\right)}{1-\alpha} b_{k_i} \leq 1. \tag{13}$$

For $\sum_{i=j}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{m,n_i} may be written as

$$\sum_{i=j}^{\infty} t_i f_{m,n_i}(z) = z - \sum_{k=j+1}^{\infty} \left(\sum_{i=j}^{\infty} t_i a_{k_i}\right) z^k + (-1)^{m+n-1} \sum_{k=j}^{\infty} \left(\sum_{i=j}^{\infty} t_i b_{k_i}\right) \overline{z}^k.$$

Then by (13),

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \frac{\left([\lambda(k-1)+1]^{m+n} - \alpha\left[\lambda(k-1)+1\right]^n\right)}{1-\alpha} \left(\sum_{i=j}^{\infty} t_i a_{k_i}\right) \\
&+ \sum_{k=j}^{\infty} \frac{\left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha\left[\lambda(k+1)-1\right]^n\right)}{1-\alpha} \left(\sum_{i=j}^{\infty} t_i b_{k_i}\right) \\
&= \sum_{i=j}^{\infty} t_i \left(\sum_{k=j+1}^{\infty} \frac{\left([\lambda(k-1)+1]^{m+n} - \alpha\left[\lambda(k-1)+1\right]^n\right)}{1-\alpha} a_{k_i} \right. \\
&\left. + \sum_{k=j}^{\infty} \frac{\left([\lambda(k+1)-1]^{m+n} - (-1)^m \alpha\left[\lambda(k+1)-1\right]^n\right)}{1-\alpha} b_{k_i} \right) \\
&\leq \sum_{i=j}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (10) and so $\sum_{i=j}^{\infty} t_i f_{m,n_i}(z) \in \overline{SH}_j(\lambda, m, n, \alpha)$.

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