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CERTAIN DIFFERENTIAL SUBORDINATIONS USING SĂLĂGEAN AND RUSCHEWEYH OPERATORS

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Abstract. In the present paper we define a new operator using the Sălăgean and Ruscheweyh operators. Denote by SR^n the Hadamard product of the Sălăgean operator S^n and the Ruscheweyh operator R^n, given by $SR^n : A \to A$, $SR^n f(z) = (S^n * R^n) f(z)$ and $A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ is the class of normalized analytic functions with $A_1 = A$. We study some differential subordinations regarding the operator SR^n.

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1. Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} :\ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$$

with $A_1 = A$ and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by

$$K = \left\{f \in A,\ \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\},$$

the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \qquad z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Definition No. 1 *(Sălăgean [4]) For $f \in A$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : A \to A$,*

$$\begin{aligned} S^0 f(z) &= f(z) \\ S^1 f(z) &= zf'(z) \\ &\ldots \\ S^{n+1} f(z) &= z\left(S^n f(z)\right)', \quad z \in U. \end{aligned}$$

Remark No. 1 *If $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.*

Definition No. 2 *(Ruscheweyh [3]) For $f \in A$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : A \to A$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= zf'(z) \\ &\ldots \\ (n+1) R^{n+1} f(z) &= z\left(R^n f(z)\right)' + nR^n f(z), \quad z \in U. \end{aligned}$$

Remark No. 2 *If $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j$, $z \in U$.*

Lemma No. 1 *(Miller and Mocanu [2]) Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z), \quad z \in U,$$

where $\alpha > 0$ *and* n *is a positive integer.*

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots, \quad z \in U$$

is holomorphic in U *and*

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in U$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

2. Main results

Definition No. 3 *[1] Let* $n \in \mathbb{N}$*. Denote by* SR^n *the operator given by the Hadamard product (the convolution product) of the Sălăgean operator* S^n *and the Ruscheweyh operator* R^n*,* $SR^n : A \to A$*,*

$$SR^n f(z) = (S^n * R^n) f(z).$$

Remark No. 3 *If* $f \in A$*,* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$*, then* $SR^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} j^n a_j^2 z^j$*.*

Theorem No. 1 *Let* g *be a convex function such that* $g(0) = 1$ *and let* h *be the function* $h(z) = g(z) + zg'(z)$*,* $z \in U$*. If* $n \in \mathbb{N}$*,* $f \in A$ *and the differential subordination*

$$\frac{1}{z} SR^{n+1} f(z) + \frac{n}{n+1} z \left(SR^n f(z)\right)'' \prec h(z), \quad z \in U \tag{2}$$

holds, then

$$\left(SR^n f(z)\right)' \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. With notation $p(z) = \left(SR^n f(z)\right)' = 1 + \sum_{j=2}^{\infty} C_{n+j-1}^{n} j^{n+1} a_j^2 z^{j-1}$ and $p(0) = 1$, we obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$,

$p(z) + zp'(z) = \frac{1}{z} SR^{n+1} f(z) + z\frac{n}{n+1} \left(SR^n f(z)\right)''$.

We have $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$, $z \in U$. By using Lemma 1 we obtain $p(z) \prec g(z)$, $z \in U$, i.e. $\left(SR^n f(z)\right)' \prec g(z)$, $z \in U$ and this result is sharp.

Theorem No. 2 *Let g be a convex function, $g(0)=1$ and let h be the function $h(z)=g(z)+zg'(z)$, $z\in U$. If $n\in\mathbb{N}$, $f\in A$ and verifies the differential subordination*

$$\left(SR^n f(z)\right)' \prec h(z), \quad z\in U, \tag{3}$$

then

$$\frac{SR^n f(z)}{z} \prec g(z), \quad z\in U$$

and this result is sharp.

Proof. For $f\in A$, $f(z)=z+\sum_{j=2}^{\infty}a_j z^j$ we have

$SR^n f(z)=z+\sum_{j=2}^{\infty}C_{n+j-1}^{n}j^n a_j^2 z^j$, $z\in U$.

Consider $p(z)=\frac{SR^n f(z)}{z}=\frac{z+\sum_{j=2}^{\infty}C_{n+j-1}^{n}j^n a_j^2 z^j}{z}=1+\sum_{j=2}^{\infty}C_{n+j-1}^{n}j^n a_j^2 z^{j-1}$.

We have $p(z)+zp'(z)=\left(SR^n f(z)\right)'$, $z\in U$.

Then $\left(SR^n f(z)\right)'\prec h(z)$, $z\in U$ becomes $p(z)+zp'(z)\prec h(z)=g(z)+zg'(z)$, $z\in U$. By using Lemma 1 we obtain $p(z)\prec g(z)$, $z\in U$, i.e. $\frac{SR^n f(z)}{z}\prec g(z)$, $z\in U$.

Theorem No. 3 *Let g be a convex function such that $g(0)=1$ and let h be the function $h(z)=g(z)+zg'(z)$, $z\in U$. If $n\in\mathbb{N}$, $f\in A$ and verifies the differential subordination*

$$\left(\frac{zSR^{n+1}f(z)}{SR^n f(z)}\right)' \prec h(z), \quad z\in U, \tag{4}$$

then

$$\frac{SR^{n+1}f(z)}{SR^n f(z)} \prec g(z), \quad z\in U$$

and this result is sharp.

Proof. For $f\in A$, $f(z)=z+\sum_{j=2}^{\infty}a_j z^j$ we have
$SR^n f(z)=z+\sum_{j=2}^{\infty}C_{n+j-1}^{n}j^n a_j^2 z^j$, $z\in U$.

Consider

$$p(z)=\frac{SR^{n+1}f(z)}{SR^n f(z)}=\frac{z+\sum_{j=2}^{\infty}C_{n+j}^{n+1}j^{n+1}a_j^2 z^j}{z+\sum_{j=2}^{\infty}C_{n+j-1}^{n}j^n a_j^2 z^j}=\frac{1+\sum_{j=2}^{\infty}C_{n+j}^{n+}j^{n+1}a_j^2 z^{j-1}}{1+\sum_{j=2}^{\infty}C_{n+j-1}^{n}j^n a_j^2 z^{j-1}}.$$

We have

$$p'(z)=\frac{\left(SR^{n+1}f(z)\right)'}{SR^n f(z)}-p(z)\cdot\frac{\left(SR^n f(z)\right)'}{SR^n f(z)}.$$

Then

$$p(z)+zp'(z)=\left(\frac{zSR^{n+1}f(z)}{SR^n f(z)}\right)'.$$

Relation (4) becomes $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$, $z \in U$ and by using Lemma 1 we obtain $p(z) \prec g(z)$, $z \in U$, i.e.

$$\frac{SR^{n+1} f(z)}{SR^n f(z)} \prec g(z), \quad z \in U.$$

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