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ON SOME PELL POLYNOMIALS

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Abstract. In this study, Pell polynomials and its some properties are investigated. Furthermore, some explicit formulae for sums of these polynomials are derived using the matrices.

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1. Introduction

It is well known that Fibonacci polynomials, $f_n(x)$, by as follows

$$f_n(x) = x\ f_{n-1}(x) + \ f_{n-2}(x), n \geq 3, \tag{1}$$

where $f_1(x) = 1$ and $f_2(x) = x$. In [5], these polynomials are studied by Jacobsthal and defined as

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), n \geq 3, \tag{2}$$

where $J_1(x) = 1$ and $J_2(x) = 1$. Lucas polynomials are defined as

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geq 2 \tag{3}$$

where $L_0(x) = 2$ and $L_1(x) = x$. It follows from the recursion relation definition can be seen that $L_n(1) = L_n$, $n \geq 0$. That is, the sum of coefficients of $L_n(x)$ is nth Lucas number. Similarly, $f_n(1) = F_n$ and, $J_n(1) = F_n$, $n \geq 0$. The some well known identities related with them are as follows;

$$2f_{n+m}(x) = \ f_n(x) L_m(x) + f_m(x) L_n(x), \tag{4}$$

$$f_{n+m}(x) = \ f_n(x)\ \ f_{m-1}(x) + \ f_{n+1}(x)\ \ f_m(x), \tag{5}$$

$$L_n(x) = \ f_{n+1}(x)\ + f_{n-1}(x), \tag{6}$$

$$\left(x^2+4\right) f_n^2(x) = L_{n+1}(x) + L_{n-1}(x), \tag{7}$$

$$f_{n+1}(x)\ f_{n-1}(x) - f_n^2(x) = (-1)^n . \tag{8}$$

Pell polynomials, $P_n(x)$, are defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \tag{9}$$

where $P_0(x) = 0$ and $P_1(x) = 1$, $n \geq 2$. Similarly, Pell-Lucas polynomials are defined by

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), n \geq 2 \tag{10}$$

where $Q_0(x) = 2$ and $Q_1(x) = 2x$.

It is note that $P_n(\frac{x}{2}) = f_n(x)$, and $Q_n(\frac{x}{2}) = L_n(x)$. Several interesting properties of polynomials $P_n(x)$ can be also written; when $P_n(x)$ is the n^{th} Pell polynomial, then for $n \geq 2$, Pell polynomials have not the same degree. The leading coefficient of $P_n(x)$ is 2^{n-1}. For n odd number, the coefficients of $P_n(x)$ are even numbers, except for constant term. For $n \in N$ even number, we say that $2x$ divides $P_n(x)$ which $x \neq 0$. For all n, $deg[P_n(x)] = n - 1$. Notice that for $n \in N$ even and $n \geq 2$, the last terms of $P_n(x)$ are nx.

n, j	0	1	2	3
0	0			
1	1			
2	2			
3	4	1		
4	8	4		
5	16	12	1	
6	32	32	6	
7	64	80	24	1
8	128	192	80	8

(11)

Let $P(n, j)$ denotes the element in row n and column j, where $j \geq 0$, $n \geq 1$. In according to this table, for $n \geq 3$ we can write the following equation;

$$P(n, 0) = 2P(n-2, 0) + P(n-1, 0). \tag{12}$$

For example, we have $P(7, 0) = 2P(5, 0) + P(6, 0) = 64$. It can be seen that every row sum in the array of coefficient in the table is a Pell number. Also, we can write the following property;

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} P(n, j) = P_n, \tag{13}$$

where P_n is the n^{th} Pell number. The relationships between $P_n(x)$ and $Q_n(x)$ can be derived by using Binet formulas. Some of them are

$$Q_n(x) = P_{n+1}(x) + P_{n-1}(x), \tag{14}$$

$$P_{2n}(x) = P_n(x)Q_n(x), \tag{15}$$

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n, \tag{16}$$

$$Q_{n+1}(x)Q_{n-1}(x) - Q_n^2(x) = 4(-1)^{n-1}(x^2+1). \tag{17}$$

2. Some Properties of Pell Polynomials

In this section we give some formulae for sums of the Pell Polynomials by using the matrices. If P_n n^{th} is the Pell polynomial, then

$$P_n(x) = \frac{\left(x+\sqrt{x^2+1}\right)^n - \left(x-\sqrt{x^2+1}\right)^n}{2\sqrt{x^2+1}}. \tag{18}$$

The generating functions for Pell and Pell-Lucas polynomials are

$$P(x,t) = \frac{1}{1-2xt-t^2}, \tag{19}$$

and

$$Q(x,t) = \frac{2(x+t)}{1-2xt-t^2}, \tag{20}$$

respectively. The n^{th} Pell polynomial can be also computed using by the matrices. In order to this, we must firstly present a theorem describing the sequence of determinants for a general tridiagonal matrix. Let $A(n)$ be a family of tridiagonal matrices as follows

$$A(n) = \begin{bmatrix} a_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & \\ & a_{3,2} & a_{3,3} & \cdots & \\ & & \cdots & \cdots & a_{n-1,n} \\ & & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}. \tag{21}$$

Theorem 1. ([2]) *The determinants of $A(n)$ matrices are*
$\det(A(1)) = a_{1,1}$
$\det(A(2)) = a_{2,2}a_{1,1} - a_{2,1}a_{1,2}$
$\det(A(n)) = a_{n,n}\det(A(n-1)) - a_{n,n-1}a_{n-1,n}\det(A(n-2))$.

By considering this theorem, we can compute the n^{th} Pell polynomial. So, we can give the next theorem.

Theorem 2. *If $D_n(x)$ is a $n \times n$ tridiagonal matrix where $D_0(x) = 0$ and*

$$D_n(x) = \begin{bmatrix} 1 & i & & & \\ 0 & 2x & i & & \\ & i & 2x & \dots & \\ & & \dots & \dots & i \\ & & & i & 2x \end{bmatrix}, \tag{22}$$

then for $n \geq 0$, $\det(D_n(x)) = P_n(x)$.

Proof. We can easily seen that $D_0(x) = 0$, $\det(D_1(x)) = 1$, $\det(D_2(x)) = 2x$. We assume that $|D_{n-1}(x)| = P_{n-1}(x)$ and $|D_{n-2}(x)| = P_{n-2}(x)$. Then from by the Theorem 1 we can write

$$|D_n(x)| = 2x\,|D_{n-1}(x)| - i^2\,|D_{n-2}(x)| = 2xP_{n-1}(x) + P_{n-2}(x) = P_n(x). \tag{23}$$

Thus, the proof is completed.

Now, let us define a matrix different from $D_n(x)$ as follows

$$D_n^*(x) = \begin{bmatrix} 2 & i & & & \\ 0 & 2x & i & & \\ & i & 2x & \dots & \\ & & \dots & \dots & i \\ & & & i & 2x \end{bmatrix}. \tag{24}$$

If $D_n^*(x)$ matrix is defined as above, then, for $n \geq 1$ we get

$$|D_n^*(x)| = Q_{n-1}(x). \tag{25}$$

If we consider the recurrence relation for Pell polynomials, then we have the following theorem.

Theorem 3. *If t is a square matrix with $t^2 = 2xt + I$, then for all $n \in Z$ we have*

$$t^n = P_n(x)\,t + P_{n-1}(x)\,I, \tag{26}$$

where $P_n(x)$ is a n^{th} Pell polynomial and I is a unit matrix.

Proof. If $n = 0$, then the proof is obvious. It can be shown that by induction that

$$t^n = P_n(x)\,t + P_{n-1}(x)\,I. \tag{27}$$

For every $n \in N$, we will show that $t^{-n} = P_{-n}(x)\,t + P_{-n-1}(x)\,I$. Let $y = 2x - t = -t^{-1}$. Then $y^2 = 2xy + I$ and $y^3 = 4x^2 y + 2x + y$. Thus, we get

$$y^n = P_n(x)\,y + P_{n-1}(x)\,I. \tag{28}$$

Therefore, we can write

$$y^n = (-1)^n t^{-n} = (-1)^n P_n(x)\,y + (-1)^n P_{n-1}(x)\,I\ ,\ y = 2x - t. \tag{29}$$

So, we obtain

$$t^{-n} = P_{-n}(x)\,t + P_{-n-1}(x)\,I. \tag{30}$$

Thus, for all $n \in Z$ we have

$$t^n = P_n(x)\,t + P_{n-1}(x)\,I. \tag{31}$$

Let us consider the following matrix for using in the next theorems. If

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}, \tag{32}$$

then, it is known that

$$P^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix}, \tag{33}$$

and $\det(P^n) = (-1)^n$. So, we can give the following theorem without proof.

Theorem 4. *For all n, $m \in Z$ we have*

$$P_{n+m}(x) = P_{n+1}(x)\,P_m(x) + P_n(x)\,P_{m-1}(x)\,, \tag{34}$$

$$Q_{n+m}(x) = Q_{n+1}(x)\,P_m(x) + Q_n(x)\,P_{m-1}(x)\,. \tag{35}$$

Furthermore, from multiplication of $P^n(x)$ and $P^m(x)$ we have

$$2P_{n+m}(x) = P_m(x)\,Q_n(x) + P_n(x)\,Q_m(x)\,. \tag{36}$$

Also, from $P_{m+n}(x)$ polynomials, we can get

$$P^2_{n+1}(x) + P^2_n(x) = P_{2n+1}(x)\,. \tag{37}$$

Corollary 1. *For all n, $m \in Z$ we have*

$$(-1)^n P_{m-n}(x) = P_m(x)\,P_{n+1}(x) - P_n(x)\,P_{m+1}(x)\,. \tag{38}$$

Proof. By the matrix properties, we can write

$$P^{m-n}(x) = P^{m}(x)\,P^{-n}(x) \tag{39}$$

and

$$P^{-n}(x) = \frac{1}{(-1)^{n}} \begin{bmatrix} P_{n-1}(x) & -P_{n}(x) \\ -P_{n}(x) & P_{n+1}(x) \end{bmatrix}. \tag{40}$$

Since,

$$(-1)^{n} P^{m-n}(x) =$$
$$= \begin{bmatrix} P_{m+1}(x)\,P_{n-1}(x) - P_{n}(x)\,P_{m}(x) & P_{n+1}(x)\,P_{m}(x) - P_{m+1}(x)\,P_{n}(x) \\ P_{m}(x)\,P_{n-1}(x) - P_{n}(x)\,P_{m-1}(x) & P_{n+1}(x)\,P_{m-1}(x) - P_{n}(x)\,P_{m}(x) \end{bmatrix} \tag{41}$$

It follows from that

$$(-1)^{n} P_{m-n}(x) = P_{m}(x)\,P_{n+1}(x) - P_{n}(x)\,P_{m+1}(x). \tag{42}$$

So, the proof is completed.

Furthermore, for all $n, m \in Z$ one can write

$$P_{n+m}(x) + (-1)^{n} P_{m-n}(x) = P_{m}(x)\,Q_{n}(x). \tag{43}$$

By the properties and definition of the $P^{n}(x)$ matrix it can be seen that $\det\left(P^{m+n}(x) + (-1)^{n} P^{m-n}(x)\right)$ is equal to

$$\left(P_{m+1}(x)\,Q_{n}(x) - P_{m}(x)\,Q_{n}(x)\right)\left(P_{m+1}(x)\,Q_{n}(x) + P_{m}(x)\,Q_{n}(x)\right). \tag{44}$$

Moreover, we can get

$$Q_{n+r}(x) + Q_{n-r}(x) = Q_{n}(x)\,Q_{r}(x). \tag{45}$$

Now, we will give a sum formula for the Pell polynomials.

Corollary 2. *For $P_{m}(x)$ and $Q_{m}(x)$, we have*

$$\sum_{m=1}^{n} P_{m}(x) = \frac{P_{n+1}(x) + P_{n}(x) - 1}{2}, \tag{46}$$

$$\sum_{m=1}^{n} Q_{m}(x) = \frac{Q_{n+1}(x) + Q_{n}(x) - 2 - 2x}{2x}. \tag{47}$$

respectively.

Theorem 5. *If $n \in N$ and $m,\ k \in Z$ with $m \neq 0$, then we have*

$$\sum_{j=0}^{n} P_{mj+k}(x) = \frac{P_m(x)\left(1-(P^m(x))^{n+1}\right)P^k(x)}{1+(-1)^m - Q_m(x)}. \tag{48}$$

Proof. It is well known that

$$\left(I-(P^m(x))^{n+1}\right) = (I-(P^m(x)))\sum_{j=0}^{n} P^{mj}(x). \tag{49}$$

If $\det(I-(P^m(x)) \neq 0$, then we have

$$((I-(P^m(x)))^{-1}(I-(P^m(x))^{n+1} = \sum_{j=0}^{n} P^{mj}(x), \tag{50}$$

and

$$((I-(P^m(x)))^{-1}(I-\left(P^{mn+m+k}(x)\right) = \sum_{j=0}^{n} P^{mj+k}(x). \tag{51}$$

Also, we can get

$$\sum_{j=0}^{n} P^{mj+k}(x) = \begin{bmatrix} \sum\limits_{j=0}^{n} P_{mj+k+1}(x) & \sum\limits_{j=0}^{n} P_{mj+k}(x) \\ \sum\limits_{j=0}^{n} P_{mj+k}(x) & \sum\limits_{j=0}^{n} P_{mj+k-1}(x) \end{bmatrix} \tag{52}$$

Since

$$A = I-(P^m(x)), A = \begin{bmatrix} 1-P_{m+1}(x) & -P_m(x) \\ -P_m(x) & 1-P_{m-1}(x) \end{bmatrix}, \tag{53}$$

and $\det(A) = 1+(-1^m) - Q_m(x)$. We have

$$A^{-1} = \frac{1}{1+(-1)^m - Q_m(x)} \begin{bmatrix} 1-P_{m-1}(x) & P_m(x) \\ P_m(x) & 1-P_{m+1}(x) \end{bmatrix}. \tag{54}$$

So, we can obtain that

$$\sum_{j=0}^{n} P_{mj+k}(x) = \frac{P_m(x)\left(1-(P^m(x))^{n+1}\right)P^k(x)}{1+(-1)^m - Q_m(x)}. \tag{55}$$

Thus, the proof is completed.

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