Acta Universitatis Apulensis No. 29/2012
ISSN: 1582-5329 pp. 31-45

SOME REMARKS ON THE ACTION OF LUSIN AREA OPERATOR IN BERGMAN SPACES OF THE UNIT BALL

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Abstract. We study the action of Lusin area operator on Bergman classes in the unit ball, providing some direct generalizations of recent results of Z. Wu.

2000 *Mathematics Subject Classification*: 32A10.

1. Introduction

Let as usual $\mathrm{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball of $\mathbb{C}^n$ and S the unit sphere plane of $\mathbb{C}^n$. Let dv be the normalized Lebesgue measure on B and $d\sigma$ the normalized rotation invariant Lebesgue measure on S. We denote by $H(\mathrm{B})$ as usual the class of all holomorphic functions on B.

For any real parameter α we consider the weighted volume measure

$$dv_\alpha(z) = (1 - |z|^2)^\alpha dv(z).$$

Suppose $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those functions $f \in H(\mathrm{B})$ for which

$$\|f\|^p_{A^p_\alpha} = \int_{\mathrm{B}} |f(z)|^p dv_\alpha(z) < \infty.$$

Let $r > 0$ and $z \in \mathrm{B}$, the Bergman metric ball at z is defined as

$$D(z,r) = \{w \in \mathrm{B} : \beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} < r\}.$$

Where the involution φ_z has the form

$$\varphi_z(w) = \frac{z - P_z(w) - s_z Q_z(w)}{1 - \langle w, z\rangle},$$

where by $s_z = (1-|z|^2)^{\frac{1}{2}}$, P_z is the orthogonal projection into the space spanned by $z \in \mathrm{B}$, i.e. $P_w(z) = \frac{\langle w,z\rangle z}{|z|^2}$, $P_0(w) = 0$ and $Q_z = I - P_z$ (see [4]). The volume of $D(z,r)$ is given by (see [4])

$$\upsilon(D(z,r)) = \frac{R^{2n}(1-|z|^2)^{n+1}}{(1-R^2|z|^2)^{n+1}},$$

where $R = \tanh(r)$. Set $|D(z,r)| = \upsilon(D(z,r))$. For $w \in D(z,r)$, $r > 0$, we have that (see, for example, [4])

$$(1-|z|^2)^{n+1} \asymp (1-|w|^2)^{n+1} \asymp |1-\langle z,w\rangle|^{n+1} \asymp |D(z,r)| \tag{1}$$

and

$$|D(z,r)|^{\alpha+n+1} \asymp v_\alpha(D(z,r)). \tag{2}$$

For any $\zeta \in \mathrm{S}$ and $r > 0$, the nonisotrpic metric ball $Q_r(\zeta)$ is defined by (see [4])

$$Q_r(\zeta) = \{z \in \mathrm{B} : |1-\langle z,\zeta\rangle|^{\frac{1}{2}} < r\}.$$

A positive Borel measure μ on B is called a $\gamma-$Carleson measure if there exists a constant $C > 0$ such that

$$\mu(Q_r(\zeta)) \leq Cr^{2\gamma} \tag{3}$$

for all $\zeta \in \mathrm{S}$ and $r > 0$.

A well-known result about the $\gamma-$Carleson measure (see [3]) is that μ is a $\gamma-$Carleson measure if and only if

$$\sup_{a\in\mathrm{B}} \int_{\mathrm{B}} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{\gamma} d\mu(z) < \infty, \ \gamma > 0. \tag{4}$$

The area operator relates to the nontangential maximal function, Littlewood-Paley operator, multipliers and tent space. It is very useful in the harmonic analysis. On the unit disk, the boundedness and compactness of the area operators was studied by Cohn and Wu respectively on the Hardy space and the weighted Bergman space (see [1,3]).

Motivated by the results of [1,3], we define the area operator on the unit ball as follows. Let μ be a positive measure on B, we define Lusin area operator

$$G_{\mu,\sigma}(f)(\xi) = \int_{\Gamma_\sigma(\xi)} |f(z)| \frac{d\mu(z)}{(1-|z|)^n}, \ \ f \in H(\mathrm{B}).$$

Here $\Gamma_\sigma(\xi)$ is the corresponding Koranyi approach region with vertex ξ on S, i.e.

$$\Gamma_\sigma(\xi) = \{z \in \mathrm{B} : |1-\langle z,\zeta\rangle| < \sigma(1-|z|^2), \, \sigma > 1\}.$$

The purpose of this paper is to study the area operator on the weighted Bergman space in the unit ball in $\mathbb{C}^n$.

All embedding theorems we prove in this paper for area operator in the ball were obtained recently by Z. Wu for n=1 (case of unit disk), as in [3] our proofs are heavily based on various properties of so-called sampling sequences or r-lattice $\{a_k\}$ in the unit ball (see [4]) and various estimates connected with Bergman metric ball in $\mathbb{C}^n$ ([4]). In next section we collect preliminaries, in final section we provide formulations and proofs of all main results of this paper.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1}B \leq A \leq CB$ and $A \lesssim B$ if there is a positive constant C such that $A < CB$.

2.Preliminaries

To state and prove our results, let's collect some nice properties of the Bergman metric ball that will be used in this paper.

Lemma 1. **([4])** *There exists a positive integer N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in B with the following properties:*
(1) $\mathrm{B} = \bigcup_k D(a_k, r)$;
(2) *The sets $D(a_k, \frac{r}{4})$ are mutually disjoint*;
(3) *Each point $z \in B$ belongs to at most N of the sets $D(a_k, 2r)$.*

Remark 1. If $\{a_k\}$ is a sequence from Lemma 1, according to the result on page 78 of [4], there exist positive constants C_1, C_2 such that

$$C_1 \int_{\mathrm{B}} |f(z)|^p dv_\alpha(z) \leq \sum_{k=1}^{\infty} |f(a_k)|^p (1 - |a_k|^2)^{n+1+\alpha} \leq C_2 \int_{\mathrm{B}} |f(z)|^p dv_\alpha(z) \tag{5}$$

for all $f \in A^p_\alpha$. Such a sequence will be called a sampling sequence or r-lattice for A^p_α.

Lemma 2. **([4])** *For every $r > 0$ there exists a positive constant C_r such that*

$$C_r^{-1} \leq \frac{1 - |a|^2}{1 - |z|^2} \leq C_r, \;\; C_r^{-1} \leq \frac{1 - |a|^2}{|1 - \langle a, z \rangle|} \leq C_r,$$

for all a and z in B such that $\beta(a, z) < r$.

Lemma 3. **([4])** *Suppose $r > 0$, $p > 0$ and $\alpha > -1$. Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all $f \in H(B)$ and $z \in B$.

Lemma 4. ([4]) *Suppose $s > 0$ and $t > -1$. Then*

$$\int_{\mathrm{S}} \frac{d\sigma(\xi)}{|1 - \langle z, \xi \rangle|^{n+s}} \asymp (1 - |z|^2)^{-s} \tag{6}$$

and

$$\int_{\mathrm{B}} \frac{(1 - |w|^2)^t d\upsilon(w)}{|1 - \langle z, w \rangle|^{n+1+t+s}} \asymp (1 - |z|^2)^{-s} \tag{7}$$

as $|z| \to 1^-$.

It is known(see, e.g. [4]) that for every δ, there exists a sampling sequence $\{a_j\}$ such that $d(a_j, a_k) > \frac{\delta}{5}$ if $j \neq k$ and

$$\sum_{k=1}^{\infty} \chi_{D(a_k, 5\delta)}(z) \leq C. \tag{8}$$

Lemma 5. *Let $\sigma > 1$, $t > 0$, $\xi \in S$, $\widetilde{\Gamma}_{\sigma}(\xi) = \{z : |1 - \overline{\xi} z| < \sigma(1 - |z|)^{\frac{1}{n}}\}$. Then there exist $\widetilde{\sigma}(\sigma, t) > 1$ such that $D(z, t) \subset \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$ for all $z \in \Gamma_{\sigma}(\xi)$.*

Proof of Lemma 5 Let $w \in D(z, t)$, $z \in \Gamma_{\sigma}(\xi)$. We will show that $w \in \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$ for some $\widetilde{\sigma} > 1$. Since $z \in \Gamma_{\sigma}(\xi)$, then $|1 - \overline{\xi} z| < \sigma(1 - |z|)$, hence

$$\begin{aligned} |1 - \langle \xi, w \rangle| &\leq |1 - \langle \xi, z \rangle| + |\langle \overline{\xi}, z \rangle - \langle \overline{\xi}, w \rangle| \\ &\leq \sigma(1 - |z|) + |z - w| \\ &\leq \sigma(1 - |w|) + (\sigma + 1)|z - w|. \end{aligned}$$

We will show $|z - w| \leq \sigma_1 (1 - |w|)^{\frac{1}{2}}$ for some $\sigma_1 > 1$. This is enough since $w \in D(z, t)$ is the same to $z \in D(w, t)$ we have by exercise 1.1 from [4]:

$$\frac{|P_w(z) - c|^2}{R^2 \sigma_1^2} + \frac{|Q_z(w)|^2}{R^2 \sigma_1} < 1. \tag{9}$$

where

$$R = \tanh(t),\ c = \frac{(1 - R^2) w}{1 - R^2 |w|^2},\ \sigma_1 = \frac{1 - |w|^2}{1 - R^2 |w|^2},$$
$$P_w(z) = \frac{\langle z, w \rangle w}{|w|^2},\ Q_w(z) = z - \frac{\langle z, w \rangle w}{|w|^2}.$$

Hence

$$|z - w| \leq C_1 \left(|z - P_w| + \left| \frac{\langle z, w \rangle w}{|w|^2} - c \right| + |c - w| \right); \tag{10}$$

$$|c-w| \leq |w|\left(1-\frac{1-R^2}{1-R^2|w|^2}\right) \leq \frac{C_2}{1-R^2}(1-|w|) = S_2. \tag{11}$$

It is enough to show

$$\left|z-\frac{\langle z,w\rangle w}{|w|^2}\right| + \left|\frac{\langle z,w\rangle w}{|w|^2}-c\right| \leq R\left(\frac{1-|w|^2}{1-R^2|w|^2}\right)^{\frac{1}{2}}\widetilde{c}(R). \tag{12}$$

Note

$$|z-w|^2 \leq C_3\left(|c-w|^2 + \left|z-\frac{\langle z,w\rangle w}{|w|^2}\right|^2 + \left|\frac{\langle z,w\rangle w}{|w|^2}-c\right|^2\right). \tag{13}$$

Hence from (9)

$$\begin{aligned} |z-w|^2 &\leq C_4(|c-w|^2 + R^2\sigma_1) \\ &\leq C_4\left(S_2(|w|,R)^2 + \frac{R^2(1-|w|^2)}{1-R^2|w|^2}\right) \\ &\leq C_4\left(S_2 + \frac{R(1-|w|^2)^{\frac{1}{2}}}{1-R^2|w|^2}\right)^2. \end{aligned}$$

Hence $|z-w| \leq C_5(1-|w|)^{\frac{1}{2}}$. Hence we complete the proof of Lemma 5.

Remark 2. For $n=1$, Lemma 5 was proved by Wu's paper in [3].

Lemma 6. *Let μ be a positive Borel measure in B. Let $D(w,t) \subset \widetilde{\Gamma}_\sigma(\xi)$, $w \in \Gamma_\tau(\xi)$, where τ, σ, ξ, t are from Lemma 5. Let $f \in H(B)$. Then*

$$\int_{\Gamma_\tau(\xi)} \frac{|f(z)|d\mu(z)}{(1-|z|)^n} \leq C\int_{\widetilde{\Gamma}_\sigma(\xi)} |f(z)| \int_{D(z,t)} d\mu(w)\frac{1}{(1-|z|)^{2n+1}}d\upsilon(z).$$

Remark 3. For $n=1$, Lemma 6 was provided in Wu's paper in [3].

Remark 4. The careful inspection of proof of Lemma 5 shows that we can find δ, $\delta > 0$ such that $D(a,\delta) \subset \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$, if $a \in \Gamma_\sigma(\xi)$, $\sigma > 1$, for some fixed $\widetilde{\sigma}$, $\widetilde{\sigma} > 1$.

Proof of Lemma 6. Since $\chi_{D(z,t)}(w) = \chi_{D(w,t)}(z)$, $z, w \in$ B, $t > 0$. Using Lemma 3, Lemma 5 and Fubini theorem, we have

$$\begin{aligned} \int_{\Gamma_\tau(\xi)} \frac{|f(z)|d\mu(z)}{(1-|z|)^n} &\leq C\int_{\Gamma_\tau(\xi)} \frac{1}{(1-|z|)^{n+1}}\left(\int_{D(z,t)} |f(w)|d\upsilon(w)\right)\frac{d\mu(z)}{(1-|z|)^n} \\ &\lesssim \int_{\Gamma_\tau(\xi)} \frac{1}{(1-|z|)^{2n+1}}\int_{\mathrm{B}} |f(w)|\chi_{D(w,t)}(z)d\upsilon(w)d\mu(z) \\ &\lesssim \int_{\widetilde{\Gamma}_\sigma(\xi)}\int_{\Gamma_\tau(\xi)} \frac{1}{(1-|z|)^{2n+1}}|f(w)|\chi_{D(w,t)}(z)d\upsilon(w)d\mu(z) \\ &\lesssim \int_{\widetilde{\Gamma}_\sigma(\xi)} \frac{1}{(1-|w|)^{2n+1}}|f(w)|\left(\int_{D(w,t)} d\mu(z)\right)d\upsilon(w). \end{aligned}$$

Hence we complete the proof of Lemma 6.

We denote by $(A^p_\alpha)_1$ the space of all holomorphic functions in the unit ball such that

$$\|f\|^p_{(A^p_\alpha)_1} = \int_{\mathrm{S}}\int_{\widetilde{\Gamma}_t(\xi)} \frac{|f(z)|^p(1-|z|)^\alpha}{(1-|z|)^n} dv(z)d\sigma(\xi) < \infty,\ n\in\mathbb{N},\ 0<p<\infty,\ \alpha>-1,$$

where

$$\widetilde{\Gamma}_t(\xi) = \{z\in \mathrm{B}:\ |1-\xi\overline{z}| < t(1-|z|)^{\frac{1}{n}},\ t>1\}$$

enlarged approach region.

Note for $n=1$, $(A^p_\alpha)_1 = A^p_\alpha$. Since for every $t>0$, $0<p<\infty$, $\alpha>-1$, $n\geq 1$, $n\in\mathbb{N}$,

$$\|f\|^p_{A^p_\alpha} \asymp \int_{\mathrm{S}}\int_{\Gamma_t(\xi)} \frac{|f(z)|^p(1-|z|)^\alpha}{(1-|z|)^n} dv(z)d\sigma(\xi)$$

where

$$\Gamma_t(\xi) = \{z\in \mathrm{B}:\ |1-\xi\overline{z}| < t(1-|z|)\},\ t>1,$$

we note that

$$\begin{aligned} \int_{\widetilde{\Gamma}_t(\xi)} |f(z)|^p dv_\alpha(z) &\leq \int_{\mathrm{B}} |f(z)|^p dv_\alpha(z) \\ &\asymp \int_{\mathrm{S}}\int_{\Gamma_t(\xi)} \frac{|f(z)|^p dv_\alpha(z)}{(1-|z|)^n} d\sigma(\xi) \\ &\leq \int_{\mathrm{S}}\int_{\widetilde{\Gamma}_t(\xi)} \frac{|f(z)|^p dv_\alpha(z)}{(1-|z|)^n} d\sigma(\xi), \end{aligned}$$

where $0<p<\infty$, $\alpha>-1$, $f\in (A^p_\alpha)_1(\mathrm{B})$. We will use this observation in the proof of Theorem 1.

We will also need Khinchine type estimate.

Remark 5. Let us remind classical Khinchine's inequality. Let $t\in[0,1)$,

$$r_0(s) = \begin{cases} 1, & 0\leq s-[s] < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq s-[s] < 1. \end{cases}$$

$r_j(t) = r_0(2^j t)$, $j=1,2,\ldots$. Classical Khinchine's inequality says:

$$C_p\left(\sum_{j=0}^N |a_j|^2\right)^{\frac{p}{2}} \leq \int_0^1 |\sum_{j=0}^N a_j r_j(t)|^p dt \leq \frac{1}{C_p}\left(\sum_{j=0}^N |a_j|^2\right)^{\frac{p}{2}},\ 0<p<\infty,$$

Where a_j are real numbers.

We will also use the following assertion.

Remark 6. Note that for any $\{z_j\}$-r-lattice in B and any large enough m and for

$$f_j(z) = \frac{(1-|z_j|)^m}{(1-\overline{z_j}\,z)^{m+\frac{\alpha+n+1}{p}}}, \quad j = 1, 2, \ldots, z \in \mathrm{B}$$

for sufficient small $r > 0$, $\widetilde{f}(z) = \sum_j a_j f_j(z)$ is in $A^p_\alpha(\mathrm{B})$ by Theorem 2.30 of [4] for any $a_j \in l^p$ and $\|\widetilde{f}\|_{A^p_\alpha} \leq C\|a_j\|_{l^p}$.

3. Main results

The goal of this section is to prove several direct generalizations of recent results of Z. Wu from [3] on the action of Luzin area operator in Bergman classes in the unit disk. We consider such an area operator based on ordinary Koranyi approach region and expanded admissible approach region that coincide with each other in case of unit disk and study it is action on Bergman classes in the unit ball.

Theorem 1. *Let μ be a positive Borel measure in B. Let $f \in (A^p_\alpha)_1$, $p \leq q < \infty$, $0 < p \leq 1$, $\alpha > -1$. Then*

$$\left\{ \int_{\mathrm{S}} \left(\int_{\Gamma_t(\xi)} \frac{|f(z)|d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi) \right\}^{\frac{1}{q}} \leq C\|f\|_{(A^p_\alpha)_1}, \text{ for some } t > 1$$

if for some $\delta > 0$, $\int_{D(z,\delta)} d\mu(w) \leq C(1-|z|)^\gamma$, where $\gamma = \frac{\alpha+n+1}{p} + n - \frac{n}{q}$. The reverse assertion is true for A^p_α class and enlarged approach region.

Remark 7. For $n = 1$, obviously enlarged approach region coinciding with ordinary approach region, moreover $A^p_\alpha = (A^p_\alpha)_1$, Theorem 1 was obtained by Z. Wu in [3].

Proof of Theorem 1 First note that from Theorem 2.25 of [4] it follows that for some $\delta > 0$ in formulation of Theorem 1 can be replaced by for any $\delta > 0$.

First we prove the second part. Let us note that the following estimate is true by Lemma 4.

$$\|f_a\|_{A^p_\alpha} \leq C, \text{ if } f_a(z) = \frac{(1-|a|)^m}{(1-\overline{a}z)^{m+\frac{\alpha+n+1}{p}}},$$

where $a \in \mathrm{B}$ is fixed, $m > 0$. Hence

$$G = \left\{ \int_{\mathrm{S}} \left(\int_{\widetilde{\Gamma}_t(\xi)} \frac{|f_a(z)|d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi) \right\}^{\frac{1}{q}} \leq C.$$

We now estimate $G = G(f_a, \mu)$ from below. For that reason we use the following estimates.

First if $z \in D(a, \delta)$, then by Lemma 2, we easily get $|f_a(z)| \asymp (1-|a|)^{-\frac{\alpha+n+1}{p}}$. On the other hand, the accurate inspection of proof of Lemma 5 (see Remark 4) shows that we can find δ, $\delta > 0$ such that $D(a,\,\delta) \subset \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$, if $a \in \Gamma_\sigma(\xi)$, $\sigma > 1$, for some fixed $\widetilde{\sigma}$, $\widetilde{\sigma} > 1$. Using all that we can estimate $G(f_a, \mu)$ from below to obtain the estimate we need. We have

$$\begin{aligned} G(f_a,\mu)^q &\geq \int_{I_\sigma(a)} \left(\int_{D(a,\delta)} \frac{|f_a(z)| d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi) \\ &\geq C(1-|a|)^{-\frac{\alpha+n+1}{p}q} \cdot (\mu(D(a,\delta)))^q \\ &\quad \cdot \frac{1}{(1-|a|)^{nq}} \cdot \int_{I_\sigma(a)} d\sigma(\xi), \end{aligned}$$

where $I_\sigma(a) = \{\xi \in \mathrm{S} : a \in \Gamma_\sigma(\xi)\}$, $\sigma > 0$, $a \in \mathrm{B}$. As it was noted in [4], the following estimate holds.

$$|I_\sigma(a)| = \int_{I_\sigma(a)} d\xi = \int_{\mathrm{S}} \chi_{\Gamma_\sigma(\xi)(a)} d\xi \asymp (1-|a|)^n.$$

Hence we have that

$$G(f_a,\mu)^q \geq C(\mu(D(a,\delta)))^q (1-|a|)^{-\frac{\alpha+n+1}{p}q - nq + n}.$$

So finally we have

$$\mu(D(a,\delta)) \leq C(1-|a|)^{\frac{\alpha+n+1}{p} + n - \frac{n}{q}},$$

for some $\delta > 0$ and all $a \in \mathrm{B}$.

For $q = \infty$, we have

$$\begin{aligned} \frac{\mu(D(a,\delta))}{(1-|a|)^{\frac{\alpha+n+1}{p}+n}} &\leq C \int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)} \frac{|f_a(z)| d\mu(z)}{(1-|z|)^n} \\ &\leq C \left\| \int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)} \frac{|f_a(z)| d\mu(z)}{(1-|z|)^n} \right\|_{L^\infty} \\ &\leq C \|f_a\|_{A^p_\alpha}. \end{aligned}$$

The condition we obtained above on measure is also sufficient. To show that we will need new estimates. For that reason we choose $\{z_j\}$ lattice ($\widetilde{\delta}$-lattice) in B with $\widetilde{\delta}$ less than δ,

$$\mu(D(z_j,\widetilde{\delta})) \leq C(1-|z_j|)^{\frac{\alpha+n+1}{p}+n-\frac{n}{q}},$$

Hence

$$\begin{aligned}S_p(f,\mu) &= \left(\int_{\Gamma_t(\xi)} \frac{|f(z)|d\mu(z)}{(1-|z|)^n}\right)^p \\ &\leq \left\{C \sum_{j,\, D(z_j,\widetilde{\delta})\bigcap\Gamma_t(\xi)\neq\varnothing} \left(\sup_{w\in D(z_j,\widetilde{\delta})}|f(w)|\right)\cdot\frac{\mu(D(z_j,\widetilde{\delta}))}{(1-|z_j|)^n}\right\}^p \\ &\leq C \sum_{j,\, D(z_j,\widetilde{\delta})\bigcap\Gamma_t(\xi)\neq\varnothing} \left(\sup_{w\in D(z_j,\widetilde{\delta})}|f(w)|^p\right)\cdot(1-|z_j|)^{\alpha+n+1-n\frac{p}{q}}.\end{aligned}$$

Furthermore $D(w,\widetilde{\delta}) \subset D(z_j,2\widetilde{\delta})$ if $w \in D(z_j,\widetilde{\delta})$ by triangle inequality for β metric $\beta=\beta(z,w)$. And by Lemma 3 and Lemma 2

$$S_p(f,\mu) < C\sum_j \int_{D(z_j,2\widetilde{\delta})} |f(w)|^p(1-|w|)^{-\frac{np}{q}}\,dv_\alpha(w).$$

Since by Lemma 3 and Lemma 2, for $w \in D(z_j,\widetilde{\delta})$

$$\begin{aligned}|f(w)|^p &\leq \frac{C}{(1-|w|)^{n+1+\alpha}}\int_{D(w,\widetilde{\delta})}|f(w)|^p dv_\alpha(w) \\ &\leq \frac{C}{(1-|z_j|)^{n+1+\alpha}}\int_{D(z_j,2\widetilde{\delta})}|f(w)|^p dv_\alpha(w).\end{aligned}$$

Since for $D(z_j,\widetilde{\delta})\bigcap\Gamma_t(\xi)\neq\varnothing$ and for $z\in D(z_j,\widetilde{\delta})\bigcap\Gamma_t(\xi)$, we have $D(z_j,2\widetilde{\delta})\subset D(z,3\widetilde{\delta})\subset\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi), \widetilde{\delta}>0$, for some $\widetilde{\sigma}>0$, by Lemma 5 then using (8) we will have

$$S_p(f,\mu) \leq C\int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)} |f(z)|^p(1-|z|)^{-\frac{np}{q}}\,dv_\alpha(z) = M(f,p,q,\alpha)$$

for some $\sigma>0$,

$$(S_p(f,\mu))^{\frac{q}{p}} \leq C(M(f,p,q,\alpha))^{\frac{q}{p}}.$$

By Hölder inequality and the observation at the end of previous section

$$(M(f,p,q,\alpha)))^{\frac{q}{p}} \leq \|f\|^{q-p}_{(A^p_\alpha)_1}\int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)}\frac{|f(z)|^p dv_\alpha(z)}{(1-|z|)^n}.$$

Hence integrating both sides by sphere S we have finally what we need.

$$\|(S_p(f,\mu))^{\frac{q}{p}}\|_{L^1(S)} \lesssim C\|f\|_{(A^p_\alpha)_1} \asymp \int_{\mathrm{S}}\int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)}\frac{|f(z)|^p dv_\alpha(z)}{(1-|z|)^n}.$$

The proof of Theorem 1 is complete.

The following result follows from Lemma 5 and also can be found in Wu's paper for $n = 1$(see Theorem 3).

Proposition 2. *Let $\alpha > -1$, $q \in (1, \infty)$, μ is a positive Borel measure in B, $\delta > 0$,*

$$\|A_\mu^\delta(f)\|_{L^q} = \left\{ \int_{\mathrm{S}} \left(\int_{\Gamma_\delta(\xi)} \frac{|f(z)|}{(1-|z|)^n} d\mu(z) \right)^q d\sigma(\xi) \right\}^{\frac{1}{q}} \leq C \sup_{z \in \mathrm{B}} |f(z)|.$$

Then

$$\left\| \int_{\widetilde{\Gamma}_\tau(\xi)} \left(\int_{D(z,\widetilde{\delta})} d\mu(w) \right) \frac{dv(z)}{(1-|z|)^{2n+1}} \right\|_{L^q(\mathrm{S})} < \infty$$

for some $\widetilde{\delta} > 0$, $\tau > 0$.

Proof By Lemma 5, for any t-lattice $\{z_j\}$ in B, we have $D(z_j,\, t) \subset \widetilde{\Gamma}_\tau(\xi)$ or all $z_j \in \Gamma_\delta(\xi)$, for some $\tau > 0$, $\xi \in$ S, $\delta > 1$, $t > 0$. Hence since $\|A_\mu^\delta(1)\|_{L^q} < \infty$, we get what we need. Indeed by (1),(2), Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&\int_{\widetilde{\Gamma}_\tau(\xi)} \left(\int_{D(z,\widetilde{\delta})} d\mu(w) \right) \frac{dv(z)}{(1-|z|)^{2n+1}} \\
\leq\ & C \sum_{j,\, D(z_j,\widetilde{\delta}) \bigcap \widetilde{\Gamma}_\tau(\xi)} \int_{D(z_j,\,\widetilde{\delta})} \left(\int_{D(z_j,2\widetilde{\delta})} d\mu(w) \right) \frac{dv(z)}{(1-|z_j|)^{2n+1}} \\
\leq\ & C \sum_j \left(\int_{D(z_j,2\widetilde{\delta})} d\mu(w) \right) \frac{|D(z_j,\, \widetilde{\delta})|}{(1-|z_j|)^{2n+1}} \\
\leq\ & C \sum_j \left(\int_{D(z_j,2\widetilde{\delta})} d\mu(w) \right) \frac{1}{(1-|z_j|)^n} \leq C A_\mu^\delta(1).
\end{aligned}$$

where at the last step we used arguments provided in proof of second part of of theorem 1 which were based on estimate (8).

The proof of Proposition 2 is complete.

The following result was obtained by Wu for $n = 1$. We follow Wu's ideas expanding his arguments to unit ball in C^n.

Theorem 3. *Suppose $\alpha > -1$, $1 < q < p \leq \infty$, $t > 1$, μ be a nonnegative Borel measure in B. Then the following is true.*

$$\left\{ \int_{\mathrm{S}} \left(\int_{\Gamma_t(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi) \right\}^{\frac{1}{q}} \leq C \|f\|_{(A_\alpha^p)_1},$$

if

$$K_{p,q} = \int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)} \left(\int_{D(z,\delta)} d\mu(w) \right)^{\frac{p}{p-1}} \frac{d\upsilon(z)}{(1-|z|)^{n+(n+1)\frac{p}{p-1}}} \in L^{\frac{(p-1)q}{p-q}}(\mathrm{S})$$

for any $\widetilde{\sigma} > 1$, $\delta > 0$.

Proof Let us consider first $p = \infty$ case. The proof follows directly from Lemma 6 that we proved above. Let $f \in H(\mathrm{B})$. Then we easily have by Lemma 6

$$\begin{aligned} M_q(f,\mu) &= \int_{\mathrm{S}} \left(\int_{\Gamma_t(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi) \\ &\leq C \int_{\mathrm{S}} \left(\int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)} \frac{|\widetilde{f}(z)| d\upsilon(z)}{(1-|z|)^{2n+1}} \right)^q d\sigma(\xi), \end{aligned}$$

where

$$|\widetilde{f}(z)| = \left(\int_{D(z,\delta)} d\mu(w) \right) \cdot |f(z)|.$$

Hence

$$M_q(f,\mu) \leq \left(\sup_{z \in \mathrm{B}} |f(z)| \right) \cdot K(\mu).$$

This is enough since $K_{\infty,q}(\mu) = K(\mu)$.

Let $q > 1$, $1 < p < \infty$. Then again using Lemma 6 and Hölder inequality, we have the following chain of estimates.

$$\begin{aligned} S(\mu,f,t) &= \int_{\Gamma_t(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \\ &\leq C \int_{\widetilde{\Gamma}_\sigma(\xi)} \frac{|f(z)| \left(\int_{D(z,\delta)} d\mu(w) \right) d\upsilon(z)}{(1-|z|)^{2n+1}} \\ &\lesssim C \left(\int_{\widetilde{\Gamma}_\sigma(\xi)} \frac{|f(z)|^p d\upsilon(z)}{(1-|z|)^n} \right)^{\frac{1}{p}} \\ &\quad \cdot \left\{ \int_{\widetilde{\Gamma}_\sigma(\xi)} \frac{1}{(1-|z|)^q} \left(\int_{D(z,\delta)} d\mu(w) \right)^{p'} \frac{d\upsilon(z)}{(1-|z|)^n} \right\}^{\frac{1}{p'}}. \end{aligned}$$

Hence again using Hölder inequality, $q = (n+1)p'$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\int_{\mathrm{S}} (S(\mu,f,t))^q d\sigma(\xi) \leq C \|f\|^q_{(A^p_\alpha)_1}.$$

The proof of Theorem 3 is complete.

The following result was also obtained by Wu for $n = 1$. We follow again Wu's ideas to expand his one-dimensional arguments to the unit ball case in C^n.

Theorem 4. *Let $\alpha > -1$, $q < p \leq \infty$, $0 < q < 1$. Let μ be a nonnegative Borel measure in B. Then the following statements are true.*

1) *For any fixed $\sigma > 0$,*

$$\left\{ \int_{\mathrm{S}} \left(\int_{\Gamma_\sigma(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi) \right\}^{\frac{1}{q}} \leq C\|f\|_{A^p_\alpha}, \qquad (M_1)$$

if

$$\int_{\mathrm{S}} \left(\sum_{j,\, z_j \in \widetilde{\Gamma}_\tau(\xi)} |a_j| \int_{D(z_j,\delta)} d\mu(w) \frac{1}{(1-|z_j|)^s} \right)^q d\sigma(\xi) \leq \|\{a_j\}\|^q_{l^p} \qquad (M_2)$$

for any $\tau > 0$, $\delta-$lattice $\{z_j\}$, $s = \frac{\alpha+n+1}{p} + n$.

2) *If (M_1) holds for enlarged Koranyi region, then (M_2) holds for some $\tau > 0$ and ordinary admissible Koranyi region.*

Remark 8. For $n = 1$, Theorem 4 was proved by Wu's theorem in [3].

Proof of Theorem 4. Let first condition (M_2) holds. As in proof of Theorem 1 (arguments in second part of proof based on estimate (8)), $D(z_j,\delta) \bigcap \Gamma_\sigma(\xi) \neq \emptyset$ implies $z_j \in \widetilde{\Gamma}_\tau(\xi)$ for some $\tau > 1$ (Note this for n=1 was proved in [3]). Hence for any function f, $f \in H(\mathrm{B})$, we have the following chain of estimates, let $D_j = D(z_j,\delta)$,

$$\begin{aligned}
\int_{\Gamma_\sigma(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} &\lesssim C \sum_{j,\, D_j \bigcap \Gamma_\sigma(\xi) \neq \emptyset} \int_{D_j} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \\
&\lesssim C \sum_j \left(\sup_{z \in D_j} |f(z)| \right) \frac{\mu(D_j)}{(1-|z_j|)^n} \\
&\lesssim C \sum_j \left(\sup_{z \in D_j} |f(z)| \right) (1-|z_j|)^{\frac{\alpha+n+1}{p}} \\
&\quad \cdot \frac{1}{(1-|z_j|)^{(\alpha+n+1)(\frac{1}{p}-1)+n}} \left(\int_{D(z_j,\delta)} d\mu(w) \right) \\
&\quad \cdot \frac{1}{(1-|z_j|)^{\alpha+n+1}} \\
&\lesssim C \sum_{j,\, z_j \in \widetilde{\Gamma}_\tau(\xi)} \sup_{z \in D_j} |f(z)| (1-|z_j|)^{\frac{\alpha+n+1}{p}} \\
&\quad \cdot \left(\int_{D(z_j,\delta)} d\mu(w) \right) \cdot \frac{1}{(1-|z_j|)^{\frac{\alpha+n+1}{p}+n}}.
\end{aligned}$$

Hence we only need to show that

$$(1-|z_j|)^{\frac{\alpha+n+1}{p}}\sup_{z\in D_j}|f(z)|$$

is in l^p and it is norm dominated by $C\|f\|_{A^p_\alpha}$. This is obvious for $A^\infty_\alpha = H^\infty$, where H^∞ is the class of all bounded analytic functions in B. For $p<\infty$, we have using Lemma 1-3

$$\sum_j (1-|z_j|)^{\alpha+n+1}\Big(\sup_{z\in D_j}|f(z)|^p\Big) \le C\sum_j \int_{D(z_j,2\delta)} |f(w)|^p dv_\alpha(w) \lesssim C\|f\|^p_{A^p_\alpha}.$$

This is what we need.

Let us show the reverse assertion in Theorem 4. We will need several auxiliary assertions for that. First a theorem from [2], on Khinchine's inequality for $K-$quasinorm $\|\cdot\|$ of $K-$qausi Banach space X, then we will need the atomic decomposition of Bergman classes A^p_α which can be found in [4] and we will need also Lemma 5.

Recall that a $K-$quasinorm $\|\cdot\|$ for quasi-Banach space X is a function from X to $[0,\infty)$ which has all the properties of a norm except that the triangle inequality is replaced by $\|x+y\| \le K(\|x\|+\|y\|)$ for all $x,y\in X$.

Theorem C. (see [2]) *Suppose* $0<\tau<s<\infty, K>0$. *There exist positive constants* C *and* C *depending only on* τ, s, K *such that for any quasi-Banach space* X *with* $K-$*quasinorm* $\|\cdot\|$, *any positive integer* n *and any* $x_1,\ldots,x_n\in X$, *the following estimates hold.*

$$\begin{aligned} C\Big(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|^\tau dt\Big)^{\frac{1}{\tau}} &\le \Big(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|^s dt\Big)^{\frac{1}{s}} \\ &\le C\Big(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|^\tau dt\Big)^{\frac{1}{\tau}}. \end{aligned}$$

Theorem D. (see [4]) *Suppose* $p>0, \alpha>-1$ *and* $l>n\max\{1,\frac{1}{p}\}+\frac{\alpha+1}{p}$. *Then there exists a sequence* $\{z_k\}$ *in* B *such that* $A^p_\alpha(B)$ *consists exactly of functions of the form*

$$f(z)=\sum_{k=1}^\infty a_k \frac{(1-|z_k|^2)^{(lp-n-1-\alpha)/p}}{(1-\langle z,z_k\rangle)^l},\ z\in \mathrm{B},\ \|a_k\|_{l^p}\asymp \|f\|_{A^p_\alpha}.$$

where a_k *belongs to the sequence space* l^p *and the series converges in the norm topology of* $A^p_\alpha(B)$.

Remark 9. The sequence z_k from Theorem D is a r-lattice, for small enough $r > 0$ (see [4]).

Now we turn to the proof of Theorem 4, expanding Wu's arguments to the unit ball.

Assume that

$$\left\{ \int_{\mathrm{S}} \left(\int_{\widetilde{\Gamma}_{\sigma(\xi)}} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi) \right\}^{\frac{1}{q}} \leq C \|f\|_{A^p_\alpha} \quad \text{for any} \quad \sigma > 0. \tag{14}$$

Let $D_\rho = \{z \in \mathrm{B} : |z| < \rho\}$, $\rho \in (0,1)$. Let $f \in H(\mathrm{B})$. Put

$$\|f\|_{\xi,\sigma,\rho} = \int_{\widetilde{\Gamma}_{\sigma}(\xi) \bigcap D_\rho} \frac{|f(z)| d\mu(z)}{(1-|z|)^n}.$$

Obviously for $q \in (0,1)$ by Theorem C we have for any $f_j \in A^p_\alpha(\mathrm{B})$

$$\left(\int_0^1 \left\| \sum_j a_j r_j(t) f_j(z) \right\|^q_{\xi,\sigma,\rho} dt \right)^{\frac{1}{q}} \asymp \int_0^1 \left\| \sum_j a_j r_j(t) f_j(z) \right\|_{\xi,\sigma,\rho} dt$$

Using Fubini's theorem and Khinchine's inequality (see also Remark 5) we have

$$\int_0^1 \left| \sum_j a_j r_j(t) f_j(z) \right| dt \asymp \left(\sum_j |a_j|^2 |f_j(z)|^2 \right)^{\frac{1}{2}}.$$

$$\int_{\mathrm{S}} \left(\int_{\widetilde{\Gamma}_{\sigma}(\xi) \bigcap D_\rho} \left(\sum_j |a_j|^2 |f_j(z)|^2 \right)^{\frac{1}{2}} \frac{d\mu(z)}{(1-|z|)^n} \right)^q d\sigma(\xi)$$
$$\asymp \int_{\mathrm{S}} \int_0^1 \left\| \sum_j a_j r_j(t) f_j(z) \right\|^q_{\xi,\sigma,\rho} dt d\sigma(\xi). (15)$$

It is easy to note that constants in equivalence relation (15) do not depend on ρ so we can pass to limit $\rho \to 1$.

Let

$$\{a_j\} \in l^p; \quad f_j(z) = \frac{(1-|z_j|)^m}{(1-\overline{z_j}\, z)^{m+\frac{\alpha+n+1}{p}}} \quad \text{with} \quad m > n+1,\ j = 1, 2, \ldots,\ z \in \mathrm{B}.$$

Then by Theorem D and Remark 6

$$f_t(z) = \sum_j a_j r_j(t) f_j(z) \in A^p_\alpha(\mathrm{B}) \quad \text{and} \quad \|f_t\|_{A^p_\alpha} \leq C \|\{a_j\}\|_{l^p},\ t \in [0,1).$$

Putting this f_t into (14) and (15) and using the fact that

$$\Big(\sum_j |a_j|^2|f_j(z)|^2\Big)^{\frac{1}{2}} \geq |a_j||f_j(z)| \asymp \frac{|a_j|}{(1-|z_j|)^{m+\frac{\alpha+n+1}{p}}},\ z_j \in D(z,\widetilde{\delta}),\ \widetilde{\delta} > 0,$$

we have

$$\int_{\mathrm{S}} \Big(\sum_{j, D_j \subset \widetilde{\Gamma}_\sigma(\xi)} \frac{|a_j|\mu(D_j)}{(1-|z_j|)^{\frac{\alpha+n+1}{p}+n}} \Big)^q d\sigma(\xi) \leq C\|\{a_j\}\|^q_{l^p}.$$

Hence we will have

$$\int_{\mathrm{S}} \Big\{ \sum_{j, z_j \in \Gamma_\tau(\xi)} |a_j| \Big(\int_{D(z_j,\delta)} d\mu(w) \Big) \frac{1}{(1-|z_j|)^{\frac{\alpha+n+1}{p}+n}} \Big\}^q d\sigma(\xi) \leq C\|\{a_j\}\|^q_{l^p}.$$

Since as it was noted before if $z_j \in \Gamma_\tau(\xi)$ for some $\tau > 0$, then $D(z_j,\delta) \subset \widetilde{\Gamma}_\sigma(\xi)$. The proof of Theorem 4 is complete.

Remark 10. It is easy to notice that the proof of the first part of Theorem 4 is true for all $p > 0$ and $q > 0$.

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