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BOUNDARIES AND PEAK POINTS FOR α-LIPSCHITZ OPERATOR ALGEBRAS

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Abstract. In a recent paper by A.A. Shokri and et al [9], a α-Lipschitz operator from a compact metric space X into a unital commutative Banach algebra B is defined. Now in this work, we determine the Shilov and Choquet boundaries and the set of peak points of α-Lipschitz operator algebras. Also we define some subalgebras of these algebras and characterize their Shilov and Choquet boundaries. Moreover, we determine the set of peak points of them.

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1. Introduction

Let (X,d) be a compact metric space with at least two elements in $\mathbb{C}$ and $(B, \| . \|)$ be a Banach space over the scaler field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). For a constant $0 < \alpha \leq 1$ and an operator $f : X \to B$, set

$$p_\alpha(f) := \sup_{s \neq t} \frac{\| f(t) - f(s) \|}{d^\alpha(s,t)}, \quad (s, t \in X),$$

which is called the Lipschitz constant of f. Define

$$L^\alpha(X,B) := \{ f : X \to B \quad : \quad p_\alpha(f) < \infty \},$$

and

$$l^\alpha(X,B) := \left\{ f : X \to B \quad : \quad \frac{\| f(t) - f(s) \|}{d^\alpha(s,t)} \to 0 \quad as \quad d(s,t) \to 0 \right\}.$$

The elements of $L^\alpha(X,B)$ and $l^\alpha(X,B)$ are called big and little α-Lipschitz operators, respectively [9]. Let $C(X,B)$ be the set of all continuous operators from X into B and for each $f \in C(X,B)$, define

$$\| f \|_\infty := \sup_{x \in X} \| f(x) \| .$$

For f, g in $C(X, B)$ and λ in $\mathbb{F}$, define

$$(f+g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x), \quad (x \in X).$$

It is easy to see that $(C(X, B), \parallel . \parallel_\infty)$ becomes a Banach space over $\mathbb{F}$ and $L^\alpha(X, B)$ is a linear subspace of $C(X, B)$. For each element f of $L^\alpha(X, B)$, define

$$\parallel f \parallel_\alpha := \parallel f \parallel_\infty + p_\alpha(f).$$

When $(B, \parallel . \parallel)$ is a Banach space, Cao, Zhang and Xu [1] proved that $(L^\alpha(X, B), \parallel . \parallel_\alpha)$ is a Banach space over $\mathbb{F}$ and $l^\alpha(X, B)$ is a closed linear subspace of $(L^\alpha(X, B), \parallel . \parallel_\alpha)$. When $(B, \parallel . \parallel)$ is a unital commutative Banach algebra, A.A. Shokri and et al [9] proved that $(L^\alpha(X, B), \parallel . \parallel_\alpha)$ is a Banach algebra over $\mathbb{F}$ under pointwise multiplication and $l^\alpha(X, B)$ is a closed linear subalgebra of $(L^\alpha(X, B), \parallel . \parallel_\alpha)$. This algebras are uniformly dens in $C(X, B)$. Also note that, if $\alpha < \beta \leq 1$, then $L^\beta(X, B) \subset l^\alpha(X, B)$.

Furthermore, Sherbert [7,8], Weaver [10], Honary [5], Ebadian and Shokri [4] studied some properties of Lipschitz algebras.

Finally, in this paper, we will study the boundaries and peak points of the $L^\alpha(X, B)$ and the some subalgebras of $L^\alpha(X, B)$.

2. Boundaries and peak points of α-Lipschitz operator algebras

Definition 2.1. *A subalgebra $\mathcal{A}$ of $C(X, B)$ which separates the points of X, contains the constants and which is Banach algebra with respect to some norm $\|.\|$, is a Banach function algebra on X.*

Definition 2.2. *Let $\mathcal{A}$ be a Banach function algebra on X. A closed subset P of X is a peak set of $\mathcal{A}$ if there exists a function $f \in \mathcal{A}$ such that $\Lambda o f = 1$ on P and $|\Lambda o f| < 1$ on $X \setminus P$, where $\Lambda \in B^*$ (B^* is the dual space of B). If $P = \{p\}$, then p is a peak point of $\mathcal{A}$.*

The set of all peak points of $\mathcal{A}$ is denoted by $S_0(\mathcal{A})$.

Definition 2.3. *Let $\mathcal{A}$ be a Banach function algebra on X. A subset E of X is a boundary for $\mathcal{A}$ if for every $f \in \mathcal{A}$, $\Lambda o f$ attains its maximum modulus on E, ($\Lambda \in B^*$).*

It is clear that every boundary contains $S_0(\mathcal{A})$.

For a Banach function algebra $\mathcal{A}$ on a compact metric space X, we define $T_\mathcal{A}$ the state space of $\mathcal{A}$ by

$$T_\mathcal{A} := \{\varphi \in \mathcal{A}^* : \quad \|\varphi\| = \varphi(1) = 1\}.$$

$T_{\mathcal{A}}$ is a weak*-compact Housdorff convex subset of the closed unit ball in $\mathcal{A}^*$. The Choquet boundary of $\mathcal{A}$ is the set of all $x \in X$ for which φ_x is an extreme point of $T_{\mathcal{A}}$, and it is denoted by $Ch(\mathcal{A})$. The closure of $Ch(\mathcal{A})$ in X is called the Shilov boundary of $\mathcal{A}$ and is denoted by $\Gamma(\mathcal{A})$. So

$$\Gamma(\mathcal{A}) = \overline{Ch(\mathcal{A})}.$$

Also by [3,6],

$$\Gamma(\mathcal{A}) = \overline{S_0(\mathcal{A})}.$$

In the sequel, we will need the following important remark due to T. G. Honary.

Remark 2.4. [5]. Let $\mathcal{A}$ be a Banach function algebra on X and $\overline{\mathcal{A}}$ be the uniform closure of $\mathcal{A}$. Then we have $Ch(\overline{\mathcal{A}}) = Ch(\mathcal{A})$ and $\Gamma(\overline{\mathcal{A}}) = \Gamma(\mathcal{A})$.

Theorem 2.5. *Let (X, d) be a compact metric space in $\mathbb{C}$, $(B, \|.\|)$ be a unital commutative Banach algebra with unit* **e**. *Then $Ch(C(X, B)) = X$.*

Proof. Let $\Lambda \in T_B$ be fixed. Define

$$R : C(X, B) \longrightarrow C(X),$$

$$R(f) = \Lambda o f.$$

It is clear that R is injective and homomorphism. Also if $g \in C(X)$ be arbitrary, then $f := g.\mathbf{e} \in C(X, B)$ and $R(f) = g$, because for every $x \in X$ we have

$$\begin{aligned} R(f)(x) &= R(g.\mathbf{e})(x) = (\Lambda o g.\mathbf{e})(x) \\ &= \Lambda(g(x)\mathbf{e}) = g(x)\Lambda(\mathbf{e}) = g(x) \times 1 = g(x). \end{aligned}$$

So R is surjective. Therefore R is a isomorphism, and so $C(X, B) \cong C(X)$. For $x \in X$, define

$$\varepsilon_x : C(X) \longrightarrow \mathbb{C},$$

$$\varepsilon_x(g) = g(x).$$

Then $\varepsilon_x \in \Phi_{C(X)}$, where $\Phi_{C(X)}$ is the character space of $C(X)$. Since X is a compact space, $\Phi_{C(X)} = exT_{C(X)}$ by [3], where $exT_{C(X)}$ is the set of extreme points of $C(X)$. So the unique representing measure for ε_x on X is δ_x (δ_x is the point mass of x), [3]. Now for every $x \in X$, we define

$$e_x : C(X, B) \longrightarrow \mathbb{C},$$

$$e_x(f) = (\Lambda o f)(x).$$

If $f \in C(X, B)$ be arbitrary, then $R(f) \in C(X)$. So there is $g \in C(X)$ such that $R(f) = g$. Thus $\Lambda o f = g$. Since $\varepsilon_x(g) = g(x)$, $\varepsilon_x(\Lambda o f) = (\Lambda o f)(x)$. Then for every $f \in C(X, B)$, we have

$$\varepsilon_x(\Lambda o f) = e_x(f).$$

Therefore the unique representing measure for e_x on X is δ_x ⊛. Now let H_x be the set of all positive measures μ on X which represent e_x. Since $C(X,B)$ separates points,

$$Ch(C(X,B)) = \{x \in X : H_x \texttt{ contains only the point mass } \delta_x\},$$

by [6]. By ⊛, H_x contains only the point mass δ_x. Then $x \in Ch(C(X,B))$ and so $X \subseteq Ch(C(X,B))$. Since $Ch(C(X,B)) \subseteq X$, $Ch(C(X,B)) = X$.

Corollary 2.6. *By remark 2.4 and Theorem 2.5 we have*

$$Ch\,(L^\alpha(X,B)) = Ch(l^\alpha(X,B)) = Ch(C(X,B)) = X,$$

and

$$\Gamma(L^\alpha(X,B)) = \Gamma(l^\alpha(X,B)) = \Gamma(C(X,B)) = X.$$

Theorem 2.7. $S_0(C(X,B)) = X$.

Proof. By [6], we have

$$S_0(C(X,B)) \subseteq Ch(C(X,B)).$$

Then by Theorem 2.5,

$$S_0(C(X,B)) \subseteq X.$$

Let $x_0 \in X$ be arbitrary. If $x_0 \notin S_0(C(X,B))$, then for every f in $C(X,B)$ we have $(\Lambda o f)(x_0) \neq 1$ or $|(\Lambda o f)(x)| \geq 1$ for $x \geq x_0$, $(\Lambda \in B^*,\ x \in X)$. Since $x_0 \in X$, $x_0 \in Ch(C(X,B))$ by Theorem 2.5. So φ_{x_0} is an extreme point of $T_{C(X,B)}$, that is

$$\varphi_{x_0}(f) = 1, \quad (f \in C(X,B)).$$

Then

$$(\Lambda o f)(x_0) = 1, \quad (\Lambda \in B^*,\ f \in C(X,B)).$$

It is a contradiction. Then $x_0 \in S_0(C(X,B))$, and so $X \subseteq S_0(C(X,B))$.

Theorem 2.8. $S_0(l^\alpha(X,B)) = X$ *for* $0 < \alpha < 1$.

Proof. It is clear that $S_0(l^\alpha(X,B)) \subseteq X$. Let $x_0 \in X$ be arbitrary, define

$$f(x) = \left(1 - \frac{d(x,x_0)}{diam\ X}\right)\mathbf{.e}, \quad (x \in X),$$

where

$$diam X = \sup\{d(x,y)\ :\quad x,y \in X\}.$$

It is easy to see that

$$f \in L^1(X,B), \quad 0 \leq \Lambda o f \leq 1, \quad (\Lambda o f)(x_0) = 1,$$

and $(\Lambda of)(x) < 1$ for $x \in X \setminus \{x_0\}$, $(\Lambda \in B^*)$. That is f is peak at $x_0 \in X$. So x_0 belongs to $S_0(L^1(X,B))$ and so $x_0 \in S_0(l^\alpha(X,B))$, $0 < \alpha < 1$. Therefore $X \subseteq S_0(l^\alpha(X,B))$.
Corollary 2.9. $S_0(L^\alpha(X,B)) = S_0(l^\alpha(X,B)) = X$.

3. Boundaries and peak points of subalgebras of α-Lipschitz operator algebras

Let (X,d) be a compact metric space in $\mathbb{C}$, and $(B, \|.\|)$ be a unital commutative Banach algebra with unit $\mathbf{e}$. We define

$$L^\alpha_A(X,B) := \{f \in L^\alpha(X,B) : \Lambda of \quad \texttt{is analytic in the interior of } X, (\Lambda \in B^*)\},$$

$$l^\alpha_A(X,B) := \{f \in l^\alpha(X,B) : \Lambda of \quad \texttt{is analytic in the interior of } X, (\Lambda \in B^*)\},$$

$$A(X,B) := \{f \in C(X,B) : \Lambda of \quad \texttt{is analytic in the interior of } X, (\Lambda \in B^*)\}.$$

We have

$$L^\alpha_A(X,B) = L^\alpha(X,B) \cap A(X,B),$$
$$l^\alpha_A(X,B) = l^\alpha(X,B) \cap A(X,B).$$

$L^\alpha_A(X,B)$ and $l^\alpha_A(X,B)$ are uniformly dens in $A(X,B)$.
Theorem 3.1. *Let $X := \{z \in \mathbb{C} : \ |z| \le 1\}$, $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and $(B, \|.\|)$ be a unital commutative Banach algebra with unit $\mathbf{e}$. Then $\Gamma(A(X,B)) = Ch(A(X,B) = \mathbb{T}$.*
Proof. If $f \in A(X,B)$ then for $\Lambda \in B^*$, $|\Lambda of|$ assumes its maximum over X at some point of $\mathbb{T}$, by the maximum modulus principle. So $\mathbb{T}$ contains $\Gamma(A(X,B))$. On the other hand, if $|\lambda| = 1$, then

$$\begin{aligned} \|(1+\bar{\lambda}z).\mathbf{e}\|^2 &= |1+\bar{\lambda}z|^2 = 1 + 2Re(\bar{\lambda}z) + |\bar{\lambda}z|^2 \\ &\le 2 + 2Re(\bar{\lambda}z) \le 4 \; . \end{aligned}$$

Equality holding iff $\bar{\lambda}z = 1$, that is $\lambda = z$. Thus if $f(z) = (1+\bar{\lambda}z).\mathbf{e}$, then $(\Lambda of)(\lambda) = 2$, $(\Lambda \in B^*)$. But $|(\Lambda of)(\alpha)| < 2$ if $\alpha \in X \setminus \{\lambda\}$. So λ is a peak point for $A(X,B)$. Hence $\mathbb{T}$ is contained in $Ch(A(X,B))$. We conclude that

$$\Gamma(A(X,B)) = Ch(A(X,B) = \mathbb{T}.$$

Corollary 3.2. *Let $X := \{z \in \mathbb{C} : \ |z| \le 1\}$, $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and*

$(B, \|.\|)$ *be a unital commutative Banach algebra with unit* $\mathbf{e}$. *Then by remark 2.4 and Theorem 3.1, we have*

$$\begin{aligned} S_0(L^\alpha_A(X,B)) &= S_0(l^\alpha_A(X,B) = \mathbb{T}, \\ Ch(L^\alpha_A(X,B)) &= Ch(l^\alpha_A(X,B) = \mathbb{T}, \\ \Gamma(L^\alpha_A(X,B)) &= \Gamma(l^\alpha_A(X,B) = \mathbb{T}. \end{aligned}$$

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