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INVERSE AND SATURATION THEOREMS FOR LINEAR COMBINATIONS OF A NEW CLASS OF LINEAR POSITIVE OPERATORS

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Abstract.The inverse and saturation theorems for the linear combinations of a new class of linear positive operators have been studied. A number of well known operators are special cases of this class of operators. The results make use of one of the Peetre's K- functionals. The analogues of inverse and saturation theorems in simultaneous approximation have also been proved.

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1. Introduction

During the past few decades a number of authors, Becker and Nessel [1], Berens and Lorentz [2], De Vore [4], Ditzian and May [5], May [8], Shapiro [12], and Timan [13] etc. have made an extensive study of the problems related to the inverse and saturation for different classes and sequences of the linear positive operators.In the present paper we study the inverse and saturation problems for the linear combinations of a new class of linear positive operators, $T_{\lambda.}$ This class includes several well-known sequences of linear positive operators as special cases [6], in particular, the Gamma operators of Muller, the Modified Post-Widder and Post-Widder operators. Let $M(IR^+)$ be the class of complex valued functions, measurable on IR^+, $M_b(IR^+)$ the subset of $M(IR^+)$ consisting of the functions essentially bounded on IR^+. Let $G \in M(IR^+)$ be a non-negative function satisfying :

(i) $G(u)$ is continuous at $u = 1$,

(ii) for each $\delta > 0, \|\chi_{\delta,1}G\|_\infty < G(1)$, and

(iii) there exist $\theta_1, \theta_2 > 0$ such that $(u^{\theta_1} + u^{-\theta_2})G(u) \in M_b(IR^+)$,

where $\chi_{\delta,x}$ is the characteristic function of $IR^+ - (x - \delta, x + \delta)$.

Let the class of all such functions G be denoted by $T(IR^+)$. For $G \in T(IR^+), \alpha \in IR, \lambda, x \in IR^+$ and $f \in M(IR^+)$, we define

$T_\lambda(f;x) = \frac{x^{\alpha-1}}{a(\lambda)} \int_0^\infty u^{-\alpha} f(u) G^\lambda(xu^{-1}) du,$ (1.1)

where $a(\lambda) = \int_0^\infty u^{\alpha-2} G^\lambda(u) du$, whenever the above integral exists. It can be easily seen that the integral (1.1) defines a class of linear positive operators.

2. Basic Definitions and Preliminary Results

Definition 1 *Let $\Omega(> 1)$ be a continuous function defined on IR^+. We call Ω a bounding function [11] for G if for each compact $K \subseteq IR^+$ there exist positive numbers λ_K and M_K such that*

$T_{\lambda_K}(\Omega; x) < M_K$, $x \in K$. It is clear that if $G \in T(IR^+)$, then $\Omega(u) = u^p + u^{-q}$ for $p, q > 0$ is a bounding function for G. The notion of a bounding function enables us to obtain results in a uniform set-up, which, at the same time , are applicable for a general $G \in T(IR^+)$.

For a bounding function Ω, we define the set

$D_\Omega = \{f : f$ is locally integrable on IR^+ and is such that $\limsup_{u\to 0} \frac{f(u)}{\Omega(u)}$ and $\limsup_{u\to\infty} \frac{f(u)}{\Omega(u)}$ exist$\}$

Definition 2 *:Let f be a continuous function on the interval $[a,b] \subseteq IR^+$ and $\delta > 0$. The p-modulus of continuity of f is defined by*

$$\omega_p(f;\delta) = \sup_{\substack{|h|<\delta \\ x,x+ph\in[a,b]}} \left| \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} f(x+jh) \right|$$

For $p = 1$, $\omega_p(f;\delta)$ is simply written as $\omega(f;\delta)$. If $\omega(f;\delta) \le M\delta^\beta, (0 < \beta \le 1)$, where M is a constant, we say that $f \in Lip_M\beta$. We define

$Lip(\beta; a, b) = \underset{M>0}{\cup} Lip_M\beta.$

$L_\infty[a,b] =\{ f : f$ is essentially bounded on $[a,b]$ $\}$,

$AC[a,b] =\{ f : f$ is absolutely continuous on $[a,b]$ $\}$,

$Lip(p,\beta; a,b) = \{ f : f^{(k)} \in AC[a,b], k = 0,1,2,...,p-1$ and $f^{(p)} \in Lip(\beta; a,b)\}$.

For $0 < \beta \le 2$ and some constant M,

$Liz(p,\beta; a,b) = \{f : \omega_{2p}(f;\delta) \le M\delta^{\beta k}, k = 1,2,...,p-1\}.$

For $p = 1, Liz(p,\beta; a,b)$ reduces to $Lip^*(1; a,b)$.

We introduce some more classes of the functions :

$T_\infty(IR^+) = \{G \in T(IR^+) : G$ is infinitely differentiable at $u = 1$ and $G''(1) \neq 0\}$

$C_0(IR^+) = \{f : f$ is continuous on IR^+ and has a compact support in $IR^+\}$

$C^k(IR^+) = \{f : f$ is k - times continuously differentiable on $IR^+\}$

$C_0^k(IR^+) = \{f : f \in C^k(IR^+)$ and f is compactly supported on $IR^+\}$

$C_b^{(m)}(IR^+) = \{f : f$ is m-times continuously differentiable and is such that $f^k, k = 0, 1, 2, ..., m$, are bounded on $IR^+\}$.

For a $G \in T_\infty(IR^+)$ and any fixed set of positive constants $\alpha_i, i = 0, 1, 2, ..., k$, following [11] the linear combination $T_{\lambda,k}$ of the operators $T_{\alpha_i\lambda}, i = 0, 1, 2, ..., k$ is defined by

$$T_{\lambda,k}(f;x) = \frac{1}{\Delta}\begin{vmatrix} T_{\alpha_0\lambda}(f;x) & \alpha_0^{-1} & \alpha_0^{-2} & .... & \alpha_0^{-k} \\ T_{\alpha_1\lambda}(f;x) & \alpha_1^{-1} & \alpha_1^{-2} & .... & \alpha_1^{-k} \\ ..... & .... & .... & .... & .... \\ ..... & .... & .... & .... & .... \\ T_{\alpha_k\lambda}(f;x) & \alpha_k^{-1} & \alpha_k^{-2} & .... & \alpha_k^{-k} \end{vmatrix}, \tag{2.1}$$

where Δ is the determinant obtained by replacing the operator column by the entries "1". Clearly there exist constants $C(j,k), j = 0, 1, 2, ...,$ such that $\sum\limits_{j=1}^{k} C(j,k) = 1$ and $T_{\lambda,k} = \sum\limits_{j=0}^{k} C(j,k) T_{\alpha_j\lambda}$.

Let $[a', b'] \subset (a, b)$. With $\zeta = \{g : g \in C_0^{2k+2},\ \text{supp } g \subset [a', b']\}$, for $f \in C_0(IR^+)$ with supp $f \subset [a', b']$, we define

$$K(\xi; f) = \inf_{g\in\zeta}\{\|f - g\| + \xi(\|g\| + \|g^{(2k+2)}\|)\},$$

where $0 < \xi < 1$ and the norms are the max-norms on $[a', b']$.

A function $f \in C_0(IR^+)$ with supp $f \subset [a', b']$ is said to belong to the intermediate space $C_0(\beta, p + 1; a', b'), (0 < \beta \leq 2)$ if

$$\|f\|_\beta = \sup_{0<\xi<1}\{\xi^{-\frac{\beta}{2}} K(\xi; f)\} < \infty.$$

For a detailed account of Peetre's K-functionals and the intermediate spaces, we refer [3]

We state the following results ([3] and [8] are referred for the details) on the spaces $C_0(\beta, p + 1; a', b'), Liz(\beta, k + 1; a', b')$ and the functionals $K(\xi; f)$ which will be used frequently in the proofs of the inverse and saturation theorems.

Lemma 1 *-Let* $0 < a < a' < a'' < b'' < b' < b < \infty$. *If* $f \in C_0(IR^+)$ *with supp* $f \subset [a'', b'']$, *then* $f \in C_0(\beta, p + 1; a', b')$ *iff* $f \in Liz(\beta, p + 1; a, b)$.

Lemma 2 *-Let* $0 < \beta < 2$ *and* $0 < a < b < \infty$. *Then, the following statements are equivalent:*

(i) $f \in Liz(\beta, p + 1; a, b)$,

(ii) (a) if $m < \beta(p+1) < m+1, (m = 0,1,2,...,2p+1), f^{(m)}$ *exists and belongs to* $Lip(\beta,(p+1)-m; a,b)$, *and*

(b) if $m+1 = \beta(p+1), (m = 0,1,2,...,2p), f^{(m)}$ *exists and belongs to* $Lip^*(1; a,b)$.

Lemma 3 - *If for* $\xi, \eta \in (0,1)$ *and a constant* M , *there holds*

$$K(\xi; f) \leq M \left| \eta^{\frac{\beta}{2}} + \frac{\xi}{\eta} \; K(\eta; f) \right|,$$

where $0 < \beta < 2$, *then, there exists a constant* M' *such that*

$$K(\xi; f) \leq M' \xi^{\frac{\beta}{2}}.$$

Throughout this paper, $\{\lambda_n : n \in IN\}$ denotes an increasing sequence of positive numbers such that

(i) $\lambda_n \to \infty$ as $n \to \infty$, and

(ii) for some constant $C > 0$, $\frac{\lambda_{n+1}}{\lambda_n} \leq C, n \in IN.$.

3. Inverse Theorems(Ordinary Approximation)

Let $K(\xi; f)$ denote the Peetre's K-functionals. We first prove :

Lemma 4 - *Let* $0 < a < a' < a'' < b'' < b' < b < \infty$. *If* $G \in T_\infty(IR^+), f \in M_b(IR^+)$, *supp* $f \subset [a'', b'']$ *and*

$$\sup_{x\in[a,b]} |T_{\lambda_n k}(f; x) - f(x)| = o(\lambda_n^{\frac{-\beta(k+1)}{2}}), \quad (n \to \infty) \tag{3.1}$$

where $0 < \beta < 2$ *and* k *is a non-negative integer, then* $f \in C_0(IR^+)$ *and for* $\lambda \geq 1$ *there holds*

$$K(\xi; f) \leq M \left| \lambda^{\frac{-\beta(k+1)}{2}} + \lambda^{k+1} \xi K(\lambda^{-(k+1)}; f) \right|, \tag{3.2}$$

where M *is a constant.*

Proof: - Due to the condition $\frac{\lambda_{n+1}}{\lambda_n} \leq C$ it is sufficient to prove (3.2) with λ replaced by λ_n where n is sufficiently large. Since $G \in T_\infty(IR^+)$, for some $\delta > 0, G(u)$ is $(2k+2)-$ times continuously differentiable on $(1-2\delta, 1+2\delta)$. Here δ can be chosen so small that $0 < 2\delta < \min\{1 - \frac{a'}{a''}, \frac{b'}{b''} - 1\}$. It is obvious that we can find a function $G^* \in C_0^{2k+2}(IR^+)$ such that

$$G^*(u) = \left\{ \begin{array}{ll} G(u), & |u-1| \leq \delta \\ 0, & u \leq \frac{a'}{a''} \quad \text{or} \quad u \geq \frac{b'}{b''} \end{array} \right\}$$

Then, if T^*_λ denotes the operator in (1.1) obtained by replacing G by G^*, in view of (3.1) we also have

$$\sup_{x\in[a,b]} \left| T^*_{\lambda_n,k}(f; x) - f(x) \right| \leq M' \lambda_n^{-\beta\frac{(k+1)}{2}}, \quad (n \to \infty) \tag{3.3}$$

where M' is some positive constant and $T^*_{\lambda_n,k}$ are the linear combinations corresponding to the operators $\mathbf{T}^*_{\lambda_n}$. Here, we notice that $T^*_\lambda(f;x) \in C_0^{2k+2}(IR^+)$ with supp $T^*_\lambda(f;x) \subset [a',b']$ for all $\lambda \in IR^+$. In view of (3.3) it is now clear that $f \in C_0(IR^+)$ and

$$K(\xi,f) \le M\lambda_n^{-\frac{\beta(k+1)}{2}} + \xi\{\left\|T^*_{\lambda_n,k}(f;x)\right\|_{C[a',b']} + \left\|T^{*(2k+2)}_{\lambda_n,k}(f;x)\right\|_{C[a',b']}\} \tag{3.4}$$

Next, we assert that for each $g \in \zeta = \{g : g \in C_0^{2k+2}(IR^+),\ supp\ g \subset [a',b']\}$ there holds the inequality

$$\left\|T^{*(2k+2)}_\lambda(g;x)\right\|_{C[a',b']} \le A_1\lambda^{k+1}\left\|g\right\|_{C[a',b']}, \tag{3.5}$$

where A_1 is a constant. We have

$$\left|T^{*(2k+2)}_\lambda(g;x)\right| \le C_1\left\|g\right\|_\infty \sum_{j=0}^{2k+2}\sum_{\upsilon=0}^{k+1-j} \lambda^{\upsilon+j}\frac{a^{**}(\lambda)}{a^*(\lambda)}T^{**}_\lambda(\left|u-1\right|^j;1), \tag{3.6}$$

where C_1 is a constant, T^{**}_λ is the operator defined by (1.1) with G replaced by G^* and α by $\alpha+j$ and $a^{**}(\lambda)$ [7] is the corresponding $a(\lambda)$.

Now, in view of (3.6) and the fact that *supp* $g \subset [a',b']$, (3.5) is clear. Also, for every $g \in \zeta$, it is clear that

$$\left\|T^{*(2k+2)}_\lambda(g;x)\right\|_{C[a',b']} \le A_2\left\|g^{(2k+2)}\right\|_{C[a',b']}, \tag{3.7}$$

where A_2 is a constant.

Using (3.5) and (3.7), for every $g \in \zeta$ we have

$$\left\|T^*_{\lambda_n,k}(f;x)\right\|_{C[a',b']} + \left\|T^{*(2k+2)}_{\lambda_n,k}(f;x)\right\|_{C[a',b']} \tag{3.8}$$
$$\le \lambda_n^{k+1}M''\left|\left\|f-g\right\|_{C[a',b']} + \lambda_n^{-(k+1)}\{\left\|g\right\|_{C[a',b']} + \left\|g^{(2k+2)}\right\|_{C[a',b']}\}\right|,$$

where M'' is a constant. Hence, by (3.4) and (3.8) with $M = max\{M',M''\}$ and for every $g \in \zeta$, we have

$$K(\xi,f) \le M\left|\lambda_n^{-\beta(k+1)} + \lambda_n^{(k+1)}\xi\left\|f-g\right\|_{C[a',b']} + \lambda_n^{-(k+1)}\{\left\|g\right\|_{C[a',b']} + \left\|g^{(2k+2)}\right\|_{C[a',b']}\}\right| \tag{3.9}$$

Taking the infimum on the right hand side of (3.9), we get (3.2). This completes the proof of the lemma.

Now, we are in position to prove the main result of this section :

Theorem 1 *Let* $G \in T_\infty(IR^+), \Omega$ *be a bounding function for* G *and* $f \in D_\Omega$. *If* $0 < p < 2k+2, k \in IN^0$ *(set of non-negative integers) and* $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, *then in the following statements, the implication (i) ⇒ (ii) ⇒ (iii) hold :*

(i) $\sup_{x\in[a_1,b_1]} \left|T_{\lambda_n,\,k}(f;x) - f(x)\right| = o(\lambda_n^{-\frac{p}{2}}), (n \to \infty),$

(ii) If $p \ne [p], f^{([p])}$ *exists and belongs to* $Lip(p-[p];a_2,b_2)$ *and if* $p = [p], f^{(p-1)}$ *exists and belongs to* $Lip^*(1;a_2,b_2\ )$;

(iii) $\sup_{x\in[a_3,b_3]} \left|T_{\lambda,k}(f;x) - f(x)\right| = O(\lambda^{-\frac{p}{2}}), (\lambda \to \infty).$

Proof: - Since $0 < p < 2k+2$, we write $p = \beta(k+1)$ for some $\beta \in (0,2)$. We first prove that $(ii) \Rightarrow (iii)$. Assuming (ii) and *Lemma* 2 $a_2 < a_2^* = a' < a_2' < a_2'' < a_3 < b_3 < b_2'' < b_2' < b' = b_2^* < b_2$ and $g_0 \in C_0^\infty(IR^+)$ be such that $g_0(u) = 1$ for u$\in [a_2'', b_2'']$ and *supp* $g_0 \subset [a_2', b_2']$. Then since $f \in Liz(\beta, k+1; a_2, b_2)$ also $f^* = fg_0 \in Liz(\beta, k+1; a_2, b_2)$ and *supp* $f^* \subset [a_2', b_2']$. Hence by *Lemma*1, $f^* \in C_0(\beta, k+1; a_2^*, b_2^*)$. Then for $x \in [a_3, b_3]$,

$$|T_{\lambda,k}(f;x) - f(x)| \leq |T_{\lambda,k}(f-f^*;x)| + |T_{\lambda,k}(f^*;x) - f^*(x)| \leq |T_{\lambda,k}(f^*;x) - f^*(x)| + B_1\lambda^{-\frac{p}{2}}, \tag{3.10}$$

where B_1 is a constant independent of λ and x.

Now, for any $g \in \zeta$ and $x \in [a_2^*, b_2^*]$, we have

$$|T_{\lambda,k}(f^*;x) - f(x)| \leq |T_{\lambda,k}(f^*-g;x)| + |T_{\lambda,k}(g;x) - g(x)| + |g(x) - f^*(x)| \leq B_2 \|f^* - g\|_{C[a_2^*,b_2^*]} + |T_{\lambda,k}(g;x) - g(x)|,$$

where B_2 is a constant. By a mean value theorem,

$$g(u) - g(x) = \sum_{j=1}^{2k+1} \frac{g^{(j)}(x)}{j!}(u-x)^j + \frac{(u-x)^{2k+2}}{(2k+2)!} g^{(2k+2)}(\xi_u)$$

for all $u \in IR^+$, where $\xi_u \in (u,x)$. Hence

$$T_{\lambda,k}(g(u);x) - g(x) = \sum_{j=1}^{2k+1} \frac{g^{(j)}(x)}{j!} T_{\lambda,k}((u-x)^j;x) + T_{\lambda,k}\left(\frac{(u-x)^{2k+2}}{(2k+2)!} g^{(2k+2)}(\xi_u);x\right) = \sum\nolimits_1 + \sum\nolimits_2 \quad \text{(say)}.$$

By the definition of $T_{\lambda,k}$,

$$\left|\sum\nolimits_1\right| \leq B_3\lambda^{-(k+1)} \sum_{j=1}^{2k+1} \left\|g^{(j)}\right\|_{C[a_2^*,b_2^*]}, \tag{3.11 a}$$

for large λ and $x \in [a_2^*, b_2^*]$.

Also,

$$\left|\sum\nolimits_2\right| \leq \frac{\left\|g^{(2k+2)}\right\|_{C[a_2^*,b_2^*]}}{(2k+2)!} \sum_{j=0}^{k} |C(j,k)|\, T_{\alpha_j\lambda}((u-x)^{2k+2};x) \leq B_4\lambda^{-(k+1)} \left\|g^{(2k+2)}\right\|_{C[a_2^*,b_2^*]}, \tag{3.11 b}$$

where B_3, B_4 are constants.

Hence if $B_5 = \max(B_3, B_4)$, we have

$$|T_{\lambda,k}(g;x) - g(x)| \leq B_5\lambda^{-(k+1)} \sum_{j=1}^{2k+1} \left\|g^{(j)}\right\|_{C[a_2^*,b_2^*]}. \tag{3.12}$$

Since, however, there exists a constant B_6 such that

$$\sum_{j=1}^{2k+1} \left\|g^{(j)}\right\|_{C[a_2^*,b_2^*]} \leq B_6\{\|g\|_{C[a_2^*,b_2^*]} + \left\|g^{(2k+2)}\right\|_{C[a_2^*,b_2^*]}\},$$

it follows from (3.10 - 3.12) that for all sufficiently large λ

$$\sup_{x\in[a_3,b_3]} |T_{\lambda,k}(f;x) - f(x)| \tag{3.13}$$

$\leq M' \left| \|f^* - g\|_{C[a_2^*,b_2^*]} + \lambda^{-(k+1)}\{\|g\|_{C[a_2^*,b_2^*]} + \left\|g^{(2k+2)}\right\|_{C[a_2^*,b_2^*]}\} + \lambda^{-\beta(k+1)} \right|$

where M' is some constant.Taking infimum over $g \in \zeta$ in (3.13) for sufficiently large λ, we have

$$\sup_{x\in[a_3,b_3]} |T_{\lambda,k}(f;x) - f(x)| \leq M' \left| \lambda^{-\frac{\beta(k+1)}{2}} + K(\lambda^{-(k+1)}; f^*) \right|. \qquad (3.14)$$

since $f^* \in C_0(\beta, k+1; a_2^*, b_2^*)$ and $a_2^* = a', b_2^* = b'$, we have

$$K(\lambda^{-(k+1)}; f^*) \leq M''\lambda^{-\beta(k+1)}, \qquad (3.15)$$

where M'' is a constant. Also, as $p = \beta(k+1)$, it follows from (3.14) - (3.15) that

$$\sup_{x\in[a_3,b_3]} |T_{\lambda,k}(f;x) - f(x)| = O(\lambda^{-\frac{p}{2}}).$$

This completes the proof of $(ii) \Rightarrow (iii)$.

To prove that $(i) \Rightarrow (ii)$ let us assume (i). If *supp* $f \subset (a_1, b_1)$ with $a = a_1, b = b_1$, we can choose a', b', a'' and b'' such that $a < a_1 = a < a' < a'' < b'' < b' < b = b_1 < \infty$ and *supp* $f \subset [a'', b'']$. By lemma 4 we obtain

$$K(\xi; f) \leq M\lambda^{-\frac{\beta(k+1)}{2}} + \lambda^{k+1}\xi K(\lambda^{-(k+1)}; f), (\lambda \geq 1).$$

Hence by Lemma 3 we have (ii).

When *suppf* $\subset (a_1, b_1)$,we proceed as follows. If a_1^*, b_1^* are such that $a_1 < a_1^* < a_2 < b_2 < b_1^* < b_1$ and $f^* = f$ on $[a_1, b_1]$ and vanishes outside it. Then, also

$$\sup_{x\in[a_1^*,b_1^*]} |T_{\lambda_n,k}(f^*;x) - f^*(x)| = o(\lambda_n^{-\frac{p}{2}}). \qquad (3.16)$$

Let us first consider the case when $0 < p < 1$. Let $g \in C_0^\infty(IR^+)$ with *suppf* $\subset [a'', b'']$ and $g(u) = 1$ for $u \in [a_2, b_2]$ where $a_1 < a_1^* < a' < a'' < b_2 < b'' < b' < b_1^* < b_1$. Then,

$$\sup_{x\in[a',b']} |T_{\lambda_n,k}(f^*g;x) - f^*(x)g(x)| \leq \sup_{x\in[a',b']} |g(x)T_{\lambda_n,k}(f^*(u) - f^*(x);x)|$$

$$+ \sup_{x\in[a',b']} |T_{\lambda_n,k}(f^*(u)(g(u) - g(x));x)|$$

$= I_1 + I_2$, (say).

By (3.16), $I_1 = o(\lambda_n^{-\frac{p}{2}})$; and by a simple computation $I_2 = o(\lambda_n^{-\frac{p}{2}})$.

Hence, with $F = f^*g$, we have

$$\sup_{x\in[a',b']} |T_{\lambda_n,k}(F;x) - F(x)| = o(\lambda_n^{-\frac{p}{2}}), \qquad (3.17)$$

from which, since *suppf* $\subset [a', b']$, it follows that $F \in Liz(\beta, k+1; a_1, b_1)$ as before, and f$\in Liz(\beta, k+1; a_2, b_2)$. Thus by Lemma 3, (ii) holds.

Next, we assume that assertion $(i) \Rightarrow (ii)$ holds when $0 < p < m - \delta$ where $0 < \delta < \frac{1}{2}$ is arbitrary and m takes one of the values of $1, 2, ..., 2k+1$. Since, for $m = 1$ the result has already been proved, if we can establish it for $m - \delta \leq p < m + 1 - 2\delta$ the proof will be over. Then, by the assumption that

$f^{(k-1)}$ exists and belongs to $Lip^*(1-\delta; a_2^*, b_2^*)$, where $[a_2^*, b_2^*] \subset (a_1, b_1)$ is any fixed interval. Let $a_2^* < a_1^* < a_1^{**} < a' < a'' < a_2 < b_2 < b'' < b' < b_1^{**} < b_1^* < b_2^*$. We choose g as before and write $F = f^* g$ after defining $f^* = f$ on $[a_2^*, b_2^*]$ and zero otherwise. Then,

$$\sup_{x\in[a',b']} |T_{\lambda_n,k}(F;x) - F(x)| \leq \sup_{x\in[a',b']} |g(x)T_{\lambda_n,k}(f^*(u) - f^*(x);x)| +$$

$$\sup_{x\in[a',b']} |T_{\lambda_n,k}((f^*(u) - f^*(x))(g(u) - g(x));x)|$$

$$+ \sup_{x\in[a',b']} |f^*(x)T_{\lambda_n,k}(g(u) - g(x);x)|$$

$$= J_1 + J_2 + J_3, \quad \text{(say)}.$$

Obviously, $J_1 = o(\lambda_n^{-\frac{p}{2}})$, $J_2 = o(\lambda_n^{-\frac{p}{2}})$ and $J_3 = o(\lambda_n^{-\frac{p}{2}})$.

Combining these estimates, we have

$$\sup_{x\in[a',b']} |T_{\lambda_n,k}(F;x) - F(x)| = o(\lambda_n^{-\frac{p}{2}}).$$

Again, since $suppf \subset [a'', b'']$, as before $F \in Liz(\beta, k+1; a_1^*, b_1^*)$ and (ii) follows. This completes the proof of the Theorem.

4. Saturation Theorems(Ordinary Approximation)

If $G \in T_\infty(IR^+), \Omega$ is a bounding function for G and $f \in D_\Omega$, the following asymptotic relation for $T_{\lambda,k}$ holds :

$$T_{\lambda,k}(f;x) - f(x) = \lambda^{-(k+1)} \sum_{i=1}^{2k+2} \frac{f^{(i)}(x)x^i}{i!} \gamma_{i,k+1} \frac{(-1)^k}{\alpha_0\alpha_1......\alpha_k} + o(\lambda^{-(k+1)}), \tag{4.1}$$

at any $x \in IR^+$ where $f^{(2k+2)}$ exists. Moreover, if $f^{(2k+2)}$ exists and is continuous on an open interval containing $[a,b]$, (4.1) holds uniformly in $x \in [a,b]$. This asymptotic formula indicates a saturation behaviour of the linear combinations $T_{\lambda,k}$. A more precise result is as follows :

Theorem 2 *Let $k \in IN^0, \Omega$ be a bounding function for G and $f \in D_\Omega$. If $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, in the following statements, the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v) \Rightarrow (vi)$, hold.*

(i) $\sup_{x\in[a_1,b_1]} |T_{\lambda_n k}(f;x) - f(x)| = o((\lambda_n^{-(k+1)}), \ (n \to \infty),$

(ii) $f^{(2k+1)} \in AC[a_2, b_2]$ *and* $f(2k+2) \in L^\infty[a_2, b_2],$

(iii) $\sup_{x\in[a_3,b_3]} |T_{\lambda,k}(f;x) - f(x)| = o(\lambda^{-(k+1)}), \ (\lambda \to \infty),$

(iv) $\sup_{x\in[a_1,b_1]} |T_{\lambda_n k}(f;x) - f(x)| = o(\lambda_n^{-(k+1)}), \ (n \to \infty),$

(v) $$f \in C^{2k+2}[a_2, b_2] \text{ and } \sum_{i=1}^{2k+2} \frac{f^{(i)}(x)x^i}{i!}\gamma_{i,k+1} = 0, \quad x \in [a_2, b_2],$$

and

(vi) $$\sup_{x\in[a_3,b_3]} |T_{\lambda,k}(f;x) - f(x)| = o(\lambda^{-(k+1)}), \quad (\lambda \to \infty).$$

Proof: Assume (i).Let $G^* \in C_0^*(IR^+) \cap T_\infty(IR^+)$ and T_λ^* denote the operator defined as before. It is clear from the*Theorem*1 that $f^{(2k+1)}$ exists and is continuous on on each closed subinterval of (a_1, b_1). Then, let $f^* \in C_0(IR^+)$ be such that $f^* = f$ on $[a_1^*, b_1^*]$ where $a_1 < a_1^* < a_2$ and $b_1 < b_1^* < b_2$. Then, we have

$$\sup_{x\in[a_2^*,b_2^*]} |T_{\lambda_n k}(f^*;x) - f^*(x)| = o(\lambda_n^{-(k+1)}) \qquad (n \to \infty),$$

where $a_1^* < a_2^* < a_2$ and $b_1^* < b_2^* < b_1$. Also, we have

$$\sup_{x\in[a_3^*,b_3^*]} \lambda_n^{k+1} |T_{\lambda_n k}((T_\lambda^*(f^*;u);x) - T_\lambda(f;x)|$$
$$= \sup_{x\in[a_2^*,b_2^*]} \lambda_n^{k+1} T_\lambda^*(T_{\lambda_n k}(f^*;u) - f^*(u);x) = o(1),$$

where $a_2^* < a_3^* < a_2$ and $b_2 < b_3^* < b_2^*$. Hence by uniformity assertion regarding (3.1), we have

$$\left\| \sum_{i=1}^{2k+2} \frac{x^i}{i!}\gamma_{i,k+1} T_\lambda^*(f^*;x) \right\|_{C[a_3^*,b_3^*]} \le M,$$

where M is a constant. Hence for all λ sufficiently large,

$$\left\| \gamma_{2k+2,k+1} T_\lambda^{*(2k+2)}(f^*;x) \right\|_{C[a_3^*,b_3^*]} \le M_1,$$

where M_1 is a constant. But $\gamma_{2k+2,k+1} \ne 0$. Hence there exists a constant M_2 such that for all λ sufficiently large, there holds

$$\left\| T_\lambda^{*(2k+2)}(f^*;x) \right\|_{C[a_3^*,b_3^*]} < M_2.$$

Thus, for all λ sufficiently large, $T_\lambda^{*(2k+2)}(f^*;x)$ are uniformly bounded and hence belong to $L^\infty[a_3^*, b_3^*]$. As $L^\infty[a_3^*, b_3^*]$ is dual of $L^1[a_3^*, b_3^*]$, by weak-compactness, there is an $h \in L^\infty[a_3^*, b_3^*]$ and sub-net $\{\lambda_i\}$ of $\{\lambda\}$ such that $T_{\lambda_i}^{*(2k+2)}(f^*;x)$ converges to h in the weak-topology. In particular, for any $g \in C_0^*(IR^+)$ with *suppg* $\subset (a_3^*, b_3^*)$, we have,

$$\int_{a_3^*}^{b_3^*} T_{\lambda_i}^{*(2k+2)}(f^*;x)g(x)dx \to \int_{a_3^*}^{b_3^*} h(x)g(x)dx, \quad (\lambda_i \to \infty).$$

But, by integration by parts,

$$\int_{a_3^*}^{b_3^*} T_{\lambda_i}^{*(2k+2)}(f^*;x)g(x)dx = \lim_{i\to\infty} \int_{a_3^*}^{b_3^*} T_{\lambda_i}^*(f;x)g^{(2k+2)}(x)dx$$
$$= \int_{a_3^*}^{b_3^*} f^*(x)g^{(2k+2)}(x)dx,$$

for every g as above. Hence, $D^{2k+2}f^*(t) = h(t)$ is a generalized function. Thus $Df^{*(2k+2)}(t) = h(t) \in L^\infty[a_3^*, b_3^*]$, implying that $f^{*(2k+1)} \in AC[a_2, b_2]$ and $f^{*(2k+2)} \in L^\infty[a_1, b_1]$.

But , $f = f^*$ on $[a_2, b_2]$ and (ii) follows.

$(ii) \Rightarrow (iii)$ is obvious.

Now, let (iv) hold. Then, proceeding as in the proof of $(i) \Rightarrow (ii)$ we have for all λ sufficiently large,

$$\sum_{i=1}^{2k+2} \frac{x^i}{i!}\gamma_{i,k+1}T_\lambda^{*(i)}(f^*; x) = 0, \quad x \in [a_3^*, b_3^*].$$

Thus, if $P(D)$ denotes the differential operator $\sum_{i=1}^{2k+2} \frac{x^i}{i!}\gamma_{i,k+1}D^i$ and $P^*(D)$ its adjoint, for any $g \in C_0^\infty(IR^+)$ with $suppg \subset (a_3^*, b_3^*)$, we have for all λ sufficiently large,

$$0 = \int_{a_3^*}^{b_3^*} P(D)T_\lambda^*(f^*; x)g(x)dx = \int_{a_3^*}^{b_3^*} T_\lambda^*(f; x)P^*(D)g(x)dx.$$

Taking limit as $\lambda \to \infty$, we obtain

$$\int_{a_3^*}^{b_3^*} f^*(x)P^*(D)g(x)dx = 0.$$

Hence, $D^{2k+2}f^* \in C[a_3^*, b_3^*]$ and $P(D)f^*(x) = 0, x \in [a_3^*, b_3^*]$, and (v) follows, since $f^* = f$ on $[a_2, b_2]$.Thus $(iv) \Rightarrow (v)$.

Lastly, $(v) \Rightarrow (vi)$ follows from the uniformity assertion for (3.1). This completes the proof of the *Theorem*.

The Inverse and Saturation Theorems for the classes of continuously differentiable functions can be obtained as follows :

Theorem 3 *Let* $m \in IN, G \in C_b^{(m)}(IR^+) \cap T_\infty(IR^+), \Omega$ *be a bounding function for* G, *and* $f \in D_\Omega$. *If* $0 < p < 2k+2, k \in IN^0$ *and* $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, *then in the following statements the implications* $(i) \Rightarrow (ii) \Rightarrow (iii)$ *hold.*

(i) *If* $f^{(m)}$ *exists on* $[a_1, b_1]$ *and*

$$\sup_{x\in[a_1,b_1]} \left|T_{\lambda_n,k}^{(m)}(f; x) - f^{(m)}(x)\right| = o(\lambda_n^{-\frac{p}{2}}), \quad (n \to \infty),$$

(ii) *If* $p \neq [p]$ *(the greatest integer not greater than* p*),* $f^{([p]+m)}$ *exists and belngs to* $Lip(p-[p]; a_2, b_2)$ *and*

(iii) If $p = [p]$, $f^{(m+p-1)}$ *exists and belongs to* $Lip^*(1; a_2, b_2)$, *and*

$$\sup_{x\in[a_3,b_3]} \left|T_{\lambda,k}^{(m)}(f; x) - f^{(m)}(x)\right| = o(\lambda^{-\frac{p}{2}}), \quad (\lambda \to \infty).$$

Proof: Assume (i). First of all, we note that an introduction of function $G^* \in C_0^\infty(IR^+) \cap T_\infty(IR^+)$ which coincides with G in a neighbourhood of $'1'$ as in the proof of *lemma*4, implies that $f^{(m)}(x)$ is continuous on each open subinterval of $[a_1, b_1]$ and moreover that

$$\sup_{x\in[a_1^*,b_1^*]} \left|T^{*(m)}_{\lambda_n k}(f;x) - f^{(m)}(x)\right| = o(\lambda_n^{-\frac{p}{2}}), \quad (n \to \infty). \tag{4.2}$$

Next, if $f^* \in C_0^{(m)}(IR^+)$ and coincides with f on $[a_2^*, b_2^*] \subset (a_1^*, b_1^*)$, it follows that

$$\sup_{x\in[a_3^*,b_3^*]} \left|T^{*(m)}_{\lambda_n k}(f;x) - f^{*(m)}(x)\right| = o(\lambda_n^{-\frac{p}{2}}), \ (n \to \infty), \tag{4.3}$$

where $a_2^* < a_3^* < a_2 < b_2 < b_3^* < b_2^*$. But here (5.2) is equivalent to

$$\sup_{x\in[a_3^*,b_3^*]} \left|T^{*}_{\lambda_n k}(u^m f^{*(m)}(u);x) - x^m f^{*(m)}(x)\right| = o(\lambda_n^{-\frac{p}{2}}), \ (n \to \infty),. \tag{4.4}$$

Thus, by *Theorem*1, since $f^* = f$ on $[a_2, b_2]$, we have (ii).

Next, assume that $f^* \in C_0^{(m)}(IR^+)$ which coincide with f on $[a_2', b_2'] \subset (a_2, b_2)$.Then $(u^m f^{*(m)})^{([p])} \in Lip(p-[p]; a_2', b_2')$, if $p \neq [p]$ and $(u^m f^{*(m)})^{(p-1)} \in Lip(1; a_2', b_2')$ if $p = [p]$. Hence, by *Theorem*1, if $a_2' < a_3' < a_3 < b_3 < b_3' < b_2'$

$$\sup_{x\in[a_3',b_3']} \left|T_{\lambda k}(u^m f^{*(m)}(u);x) - x^m f^{*(m)}(x)\right| = o(\lambda^{-\frac{p}{2}}),$$

$(\lambda \to \infty)$.

But, this is equivalent to

$$\sup_{x\in[a_3',b_3']} \left|T^{(m)}_{\lambda k}(f^*(u);x) - f^{*(m)}(x)\right| = o(\lambda^{-\frac{p}{2}}), \ (\lambda \to \infty). \tag{4.5}$$

Again, by the coincidence of f^* and g on $[a_2', b_2']$ and (4.5) we have (iii).

This completes the proof of the *Theorem*.

Theorem 4 *Let $m \in IN, k \in IN^0, G \in C_b^{(m)}(IR^+) \cap T_\infty(IR^+), \Omega$ be a bounding function for G, and $f \in D_\Omega$.If $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$, in the following statements the following implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v) \Rightarrow (vi)$ hold.*

(i) $f^{(m)}$ *exists on* $[a_1, b_1]$ *and*

$$\sup_{x\in[a_1,b_1]} \left|T^{(m)}_{\lambda_n k}(f;x) - f^{(m)}(x)\right| = o(\lambda_n^{-(k+1)}), \quad (n \to \infty),$$

(ii) $f^{(2k+m+1)} \in AC[a_2, b_2]$ *and* $f^{(2k+m+2)} \in L^\infty[a_2, b_2]$,

(iii) $$\sup_{x\in[a_3,b_3]} \left|T^{(m)}_{\lambda k}(f;x) - f^{(m)}(x)\right| = o(\lambda^{-(k+1)}), \quad (\lambda \to \infty),$$

(iv) $f^{(m)}$ *exists on* $[a_1, b_1]$ *and*

$$\sup_{x\in[a_1,b_1]} \left|T^{(m)}_{\lambda_n k}(f;x) - f^{(m)}(x)\right| = o(\lambda_n^{-(k+1)}), \quad (n \to \infty),$$

$(v) \quad f \in C^{2k+m+2}[a_2, b_2] \; and \; \sum_{i=1}^{2k+2} (\frac{f^{(i)}(x)x^i}{i!})^{(m)} \gamma_{i,k+1} = 0, x \in [a_2, b_2],$

$(vi) \quad \sup_{x \in [a_3, b_3 \;]} \left| T_{\lambda k}^{(m)}(f; x) - f^{(m)}(x) \right| = o(\lambda^{-(k+1)}), \quad (\lambda \to \infty).$

Proof: The proof of this theorem follows along the similar lines, with some essential modifications as in the case of Theorems 2 and 3 .

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References

[1] Becker, M. and Nessel, R. J., *Inverse results via smoothing*, Proc. International Conf. Constructive Function Theory, Sofia (1978).

[2] Berens, H. and Lorentz, G. G., *Inverse theorems for Bernstein polynomials*, Indiana Univ. Math. J. 21 (1972) 693-708.

[3] Butzer, P. L. and Berens H., *Semigroups of Operators and Approximation*, Springer-Verlag, New York (1967).

[4] Devore, R. A., *Saturation of positive convolution operators*, J. Approx. Theory 3(1970) 410-429.

[5] Ditzian, Z. and May C. P. *A saturation result for combinations of Bernstein polynomials*, Tohoku Math. J. 28(1976).

[6] Kunwar, B., *Approximation of analytic functions by a class of linear positive operators*, J. Approx. Theory 44, 173-182 (1985).

[7] Kunwar, B. and Bramha Dutta Pandey, *Simultaneous approximation by a class of linear positive operators*, Acta Universitatis Apulensis Romania (24/2010) 73-87.

[8] May, C. P., *Saturation and inverse theorems for combinations of a class of exponential-type operators*, Canadian Journal of Math.(1976) 1224-1250.

[9] Neeraj Kumar, *Rate of convergence for a general sequence of Durrmeyer-Type operators*. JIPAM, vol. 5, Issue 3,Art.79 (2004).

[10] Rathore, R. K. S., *Linear combinations of linear positive operators and generating relations in special functions*, Thesis IIT Delhi India(1973).

[11] Rathore, R. K. S.*Approximation of Unbounded functions with Linear Positive Operators*, Doctoral Dissertation, Technische Hogeschool Delft (1974).

[12] Shapiro, S. H., *Smoothing and Approximation of Functions*, Van Nostrand Riehold Co. New York (1969).

[13] Timan, A. F., *Theory of Approximation of Functions of a Real Variable*, Pergaman-Press, Oxford (Trans. 1963).

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