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SPACES OF STRONGLY ALMOST SUMMABLE DIFFERENCE SEQUENCES

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Abstract. The purpose of this paper is to introduce the concept of $\triangle^m_{v^\lambda}$ strongly almost convergence with respect to a sequence of moduli and $\triangle^m_{v^\lambda}$- Almost Statistical Convergence and give some relations between these two kinds of convergence.

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1. Introduction

Let ω denote the set of all real sequences $x = (x_k)$. Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ normed by as usual by $||x||_\infty = \sup_k |x_k|$. Kizmaz [19] defined the sequence spaces :

$$l_\infty(\triangle) = \{x = (x_k) : (\triangle x_k) \in l_\infty\},$$
$$c(\triangle) = \{x = (x_k) : (\triangle x_k) \in c\},$$

and

$$c_0(\triangle) = \{x = (x_k) : (\triangle x_k) \in c_0\},$$

where $\triangle x_k = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

Difference sequence spaces have been studied by Colak and Et [1], Et [6,7], Et and Esi [8], Vakeel A. Khan [14,15,16,17] and many others.

A sequence $x \in l_\infty$ is said to be almost convergent [23] if all Banach limits of x coincide. Lorentz [23] defined that

$$[c^\wedge] = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+j} \text{ exists, uniformly in j} \right\}.$$

Many authors including Lorentz [23] , Duran [4], and King [18] have studied almost convergent sequence spaces.Maddox [24,26] has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+j} - L| = 0, \quad \text{uniformly in j.}$$

By $[c^\wedge]$, we denote the space of all strongly almost convergent sequences. It is easy to see that

$$c \subset [c^\wedge] \subset c^\wedge \subset l_\infty.$$

The space of strongly almost convergent sequences was generalized by Nanda [27,28].

Let $p = (p_k)$ be a sequence of strictly positive real numbers . Nanda [27] defined

$$[c^\wedge, p] = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+j} - L|^{p_k} = 0, \quad \text{uniformly in j} \right\},$$

$$[c^\wedge, p]_0 = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+j}|^{p_k} = 0, \quad \text{uniformly in j} \right\},$$

$$[c^\wedge, p]_\infty = \left\{ x \in w : \sup_{n,j} \frac{1}{n} \sum_{k=1}^{n} |x_{k+j}|^{p_k} < \infty, \quad \text{uniformly in j} \right\}.$$

Let $\lambda = (\lambda_n)$ be a non decereasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized de la Vallee - Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ) - summable to a number l (see [22]) if $t_n(x) \to l$ as $n \to \infty$.We write

$$[V, \lambda]^0 = \left\{ x = (x_i) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_i| = 0 \right\},$$

$$[V, \lambda] = \left\{ x = (x_i) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_i - le| = 0, \text{for some } l \in \mathbb{C} \right\},$$

and

$$[V, \lambda]^\infty = \left\{ x = (x_i) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_i| < \infty \right\},$$

For the sets of sequences that are strongly summable to zerto , strongly summable and strongly bounded by the de la Vallee - poussin mathod.In the special case when $\lambda_n = n$ for $n = 1, 2, 3, \cdots$ the sets $[V, \lambda]^0$, $[V, \lambda]$ and $[V, \lambda]^\infty$ reduce the sets w_0 , w and w_∞ introduced and studied by Maddox [25].
The concept of statistical convergence was first introduced by Fast [9] and also Schoenberg [31] for real and complex sequences. Further this concept was studied by Salat [30], Fridy [11], Fridy and C.Orhan [12],Connor [2], Connor,Fridy, and Kline [3], and many others.
Let $\mathbb{N}$ and $\mathbb{C}$ be the set of natural numbers and complex numbers , respectively . If $E \subseteq \mathbb{N}$, then the natural densityof E (see Freedman and Sember [10]) is denoted by

$$\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|,$$

where the vertical bars denote the cardinality of the enclosed set . The sequence x is said to be statistically convergent to L , denoted by $stat - \lim x = L$, if for every $\epsilon > 0$, the set

$$\{k : |x_k - L| \geq \epsilon, \}$$

has natural density zero. In this case we write $stat - \lim x_k = L$.

Let $X, Y \subset \ell^0$. Then we shall write

$$M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \ell^0 : ax \in Y \text{ for all } x \in X\}.$$

The set

$$X^\alpha = M(X, l_1)$$

is called Köthe - Toeplitz dual space or α - dual of X(see [8]).
Let X be a sequence space . Then X is called
(i) Solid (or normal), if $(\alpha_k x_k) \in X$, whenever $(x_k) \in X$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.
(ii) Symmetric, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.
(iii)Perfect if $X = X^{\alpha\alpha}$.
(iv) Sequence algebra if $x.y \in X$, whenever $x, y \in X$.
It is well known that if X is perfect then X is normal (see [13]).
A function $f : [0, \infty) \to [0, \infty)$ is called a modular if

1. $f(t) = 0$ if and only if $t = 0$,
2. $f(t + u) \leq f(t) + f(u)$ *for all* $t, u \geq 0$,
3. f is increasing, and
4. f is continuous from the right of 0.

Let X be a sequence space. Then the sequence space $X(f)$ is defined as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}$$

for a modulus f([25],[29]). Kolk[20],[21] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ i.e.

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$$

2. Main Results

Let $F = (f_k)$ be a sequence of moduli, $p = (p_k)$ be a sequence of positive real numbers and $v = (v_k)$ be any fixed sequence of non zero complex numbers and $m \in \mathbb{N}$ be fixed(see [5]). This assumption is made throughout the rest of this paper. Now we define the following sequence spaces :

$$[V, \triangle^m_{v^\lambda}, F, p] = \left\{ x \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(|\triangle^m_v x_{k+j} - L)|)]^{p_k} = 0, \right.$$
$$\text{uniformly in j,for some L>0,}$$

$$[V, \triangle^m_{v^\lambda}, F, p]^0 = \left\{ x \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(|\triangle^m_v x_{k+j})|)]^{p_k} = 0, \quad \text{uniformly in j} \right\},$$

$$[V,\triangle^m_{v^\lambda},F,p]^\infty=\left\{x\in\omega:\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}[f_k(|\triangle^m_v x_{k+j})|)]^{p_k}<\infty,\quad\text{uniformly}\quad\text{in}\quad\text{j}\quad\right\},$$

where

$$\triangle^0_v x_k=(v_kx_k),\quad \triangle_v x_k=(v_kx_k-v_{k+1}x_{k+1}),\quad \triangle^m_v x_k=(\triangle^{m-1}_v x_k-\triangle^{m-1}_v x_{k+1})$$

. and so that

$$\triangle^m_v x_k=\sum_{i=0}^m(-1)^i\begin{bmatrix}m\\ i\end{bmatrix}\quad v_{k+i}x_{k+i}.$$

$\triangle^m_{v^\lambda}$- Almost Statistical Convergence

We define the following definition :

Definition . A sequence $x=(x_k)$ is is said to be $\triangle^m_{v^\lambda}$ - almost statistically convergent to the number L provided that for every $\epsilon>0$,

$$\lim_n\frac{1}{\lambda_n}|\{k\in I_n:|\triangle^m_{v^\lambda}x_{k+j}(x)-L|\geq\epsilon\}|=0,\ \text{uniformly}\quad\text{in j.}$$

In this case we write $S(\triangle^m_{v^\lambda}))-\lim x=L$ or $x_k\to LS(\triangle^m_{v^\lambda})$ and In the case $\lambda_n=n$ we shall write $S(\triangle^m_v)$ instaed of $S(\triangle^m_{v^\lambda})$.

Theorem 2.1. *Let $F=(f_k)$ be a sequence of moduli, then $[V,\triangle^m_{v^\lambda},F,p]$, $[V,\triangle^m_{v^\lambda},F,p]^0$ and $[V,\triangle^m_{v^\lambda},F,p]^\infty$ are linaer spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Omitted.

Theorem 2.2. *Let $F=(f_k)$ be a sequence of moduli, then*

$$[V,\triangle^m_{v^\lambda},F,p]^0\subset[V,\triangle^m_{v^\lambda},F,p]\subset[V,\triangle^m_{v^\lambda},F,p]^\infty.$$

Proof. Omitted.

Theorem 2.3. *The sequence spaces $[V,\triangle^m_{v^\lambda},F,p]$, $[V,\triangle^m_{v^\lambda},F,p]^0$ and $[V,\triangle^m_{v^\lambda},F,p]^\infty$ are not solid for $m\geq 1$.*

Proof. Let $p_k=1$ for all k, $F(x)=x$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $(x_k)=(k^r)\in[V,\triangle^m_{v^\lambda},F,p]^\infty$ but $(\alpha_kx_k)\notin[V,\triangle^m_{v^\lambda},F,p]^\infty$ when $\alpha_k=(-1)^k$ for all $k\in\mathbb{N}$. Hence $[V,\triangle^m_{v^\lambda},F,p]^\infty$ is not solid. The other cases can be proved by considering similar examples.

Corollary 1. *The sequence spaces $[V,\triangle^m_{v^\lambda},F,p]$, $[V,\triangle^m_{v^\lambda},F,p]^0$ and $[V,\triangle^m_{v^\lambda},F,p]^\infty$*

are not perfect for $m \geq 1$.
*Proof.*Omitted.
Theorem 2.4. *The sequence spaces* $[V, \triangle_{v^\lambda}^m, F, p]$, $[V, \triangle_{v^\lambda}^m, F, p]^0$ *and* $[V, \triangle_{v^\lambda}^m, F, p]^\infty$ *are not symmetric for* $m \geq 1$.
Proof. Let $p_k = 1$ for all k, $F(x) = x$ and $\lambda_n = n$ for all $n \in I\!\!N$. Then $(x_k) = (k^r) \in [V, \triangle_{v^\lambda}^m, F, p]^\infty$. Let (y_k) be a rearrangement of (x_k) , which is defined as follows :

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_16, x_6, x_25, x_7, x_36, x_8, x_49, x_10, \cdots\}.$$

Then $(y_k) \notin [V, \triangle_{v^\lambda}^m, F, p]^\infty$.

Remark 1. The space $[V, \triangle_{v^\lambda}^m, F, p]^0$ is not symmetric for $m \geq 2$.
Theorem 2.5. *The sequence spaces* $[V, \triangle_{v^\lambda}^m, F, p]_z$, *where* z *will denote any one of the notion* $0, 1$ *or* ∞ *are not sequence algebras.*
Proof. Let $p_k = 1$ for all $k \in I\!\!N$, $F(x) = x$ and $\lambda_n = n$ for all $n \in I\!\!N$. Then $x = (k^{m-2})$, $y = (k^{m-2}) \in [V, \triangle_{v^\lambda}^m, F, p]_z$, but $x, y \in [V, \triangle_{v^\lambda}^m, F, p]_z$.
Theorem 2.6. *Let* $\lambda = (\lambda_n)$ *be a non - decreasing sequence of positive numbers tending to* ∞ *, then*

(i) If $x_k \to L[V, \triangle_{v^\lambda}^m, F, p] \Rightarrow x_k \to LS(\triangle_{v^\lambda}^m)$,

(ii) If $x \in l_\infty(\triangle_v^m)$ and $x_k \to LS(\triangle_{v^\lambda}^m)$, then $x_k \to L[V, \triangle_{v^\lambda}^m, F, p]$,

(iii) $S(\triangle_{v^\lambda}^m) \cap l_\infty(\triangle_v^m) = [V, \triangle_{v^\lambda}^m, F, p] \cap l_\infty(\triangle_v^m)$.

Theorem 2.7. *Let* $F = (f_k)$ *be a sequence of moduli, and* $\sup_k(p_k) = H$ *. Then* $[V, \triangle_{v^\lambda}^m, F, p] \subset S(\triangle_{v^\lambda}^m)$.
Proof. Let $x \in [V, \triangle_{v^\lambda}^m, F, p]$ and $\epsilon > 0$ be given . Let $\sum_1$ denote the sum over $kleqn$ such that $|\triangle_v^m x_{k+m} - L| \geq \epsilon$ and $\sum_2$ denote the sum over $k \leq n$ such that $|\triangle_v^m x_{k+m} - L| < \epsilon$. Then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(|\triangle_v^m x_{k+j} - L|)]^{p_k}$$

$$= \frac{1}{\lambda_n} \sum_1 [f_k(|\triangle_v^m x_{k+j} - L|)]^{p_k} + \frac{1}{\lambda_n} \sum_2 [f_k(|\triangle_v^m x_{k+j} - L|)]^{p_k}$$

$$\geq \frac{1}{\lambda_n} \sum_1 [f_k(|\triangle_v^m x_{k+j} - L|)]^{p_k}$$

$$\geq \frac{1}{\lambda_n} \sum_1 [f_k(\epsilon)]^{p_k}$$

$$\geq \frac{1}{\lambda_n}\sum_1 \min([f_k(\epsilon)])^{\inf p_k}[f_k(\epsilon)])^H)$$

$$\geq \frac{1}{\lambda_n}|\{k\in I_n : |\triangle_v^m x_{k+j}-L|\geq\epsilon\}|\min([f_k(\epsilon)])^{\inf p_k}[f_k(\epsilon)])^H).$$

Hence $x \in S(\triangle_{v\lambda}^m)$.

Theorem 2.8. *Let $F=(f_k)$ be a sequence of bounded moduli, and $0<h=\inf_k(p_k)\leq p_k\leq\sup_k(p_k)=H<\infty$. Then $S(\triangle_{v\lambda}^m)\subset[V,\triangle_{v\lambda}^m,F,p]$.*

Proof. Suppose that $F=(f_k)$ is a sequence of bounded moduli. Let $\epsilon>0$ and let $\sum_1$ denote the sum over $kleqn$ such that $|\triangle_v^m x_{k+m}-L|\geq\epsilon$ and $\sum_2$ denote the sum over $k\leq n$ such that $|\triangle_v^m x_{k+m}-L|<\epsilon$. Since $F=(f_k)$ is a sequence of bounded moduli there exists an integer K such that $F(x)<K$ for all $x\geq 0$. Then

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}[f_k(|\triangle_v^m x_{k+j}-L|)]^{p_k}\\
&=\frac{1}{\lambda_n}\sum_1[f_k(|\triangle_v^m x_{k+j}-L|)]^{p_k}+\frac{1}{\lambda_n}\sum_2[f_k(|\triangle_v^m x_{k+j}-L|)]^{p_k}\\
&\leq\frac{1}{\lambda_n}\sum_1\max(K^h,K^H)+\frac{1}{\lambda_n}\sum_2[f_k(\epsilon)]^{p_k}\\
&\leq\max(K^h,K^H)\frac{1}{\lambda_n}|\{k\in I_n:|\triangle_v^m x_{k+j}-L|\geq\epsilon\}|\\
&\quad+\max(f_k(\epsilon)^h,f_k(\epsilon)^H).
\end{aligned}$$

Hence $x\in[V,\triangle_{v\lambda}^m,F,p]$.

Theorem 2.9. *Let $F=(f_k)$ be a sequence of bounded moduli, and $0<h=\inf_k(p_k)\leq p_k\leq\sup_k(p_k)=H<\infty$. Then $S(\triangle_{v\lambda}^m)=[V,\triangle_{v\lambda}^m,F,p]$ if and only if $F=(f_k)$ is a sequence of bounded moduli.*

Proof. Let $F=(f_k)$ be a sequence of bounded moduli. By Theorem 7 and Theorem 8 we have $S(\triangle_{v\lambda}^m)=[V,\triangle_{v\lambda}^m,F,p]$.

Conversely, suppose that $F=(f_k)$ is a sequence of unbounded moduli. Then there exists a positive sequence (t_k) with $f_k(t_k)=k^2$, for $k=1,2,\cdots$. If we choose

$$\triangle_v^m x_i=\begin{cases} t_k, & i=k^2,\ (i=1,2,\cdots)\\ 0, & \text{otherwise}\end{cases}$$

Then

$$\frac{1}{\lambda_n}|\{k\in I_n:|\triangle_v^m x_{k+j}|\geq\epsilon\}|\leq\frac{\sqrt{\lambda_{n-1}}}{\lambda_n}\quad\text{for all } n \text{ and } j.$$

This implies that $x \in S(\triangle_{v\lambda}^{m})$, but $x \notin [V, \triangle_{v\lambda}^{m}, F, p]$.This contradicts to $S(\triangle_{v\lambda}^{m}) = [V, \triangle_{v\lambda}^{m}, F, p]$.

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