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# ON MULTIVALUED CARISTI TYPE FIXED POINT THEOREMS

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Abstract In this paper, we prove some multivalued Caristi type fixed point theorems. These results generalize the corresponding generalized Caristi's fixed point theorems due to Kada-Suzuki-Takahashi (1996), Bae (2003), Suzuki (2005), Khamsi (2008) and others.

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## 1. Introduction

In 1976, Caristi [3] proved very interesting fixed point theorem on complete metric spaces, which is a genalizatrion of the well-known Banach contraction principle. The Caristi's fixed point theorem, equivalent to Ekland variational principle [4], is an important tool in nonlinear analysis and has extensive applications in the fields of variational inequalities, optimization, control theory and differential equations. Many authors have studied and generalized Caristi's fixed point theorem to various directions. Kada et al. [7] introduced the concept of $w$-distance on metric space and improved single-valued Caristi's fixed point theorem. Recently, generalizing the concept of $w$-distance, Suzuki [10] introduced the concept of $\tau$-distance on metric spaces and proved Caristi's fixed point theorem for singlevalued maps with respect to $\tau$-distance.

In this note, we prove some multivalued Caristi type fixed point theorem with respect to $\tau$-distance which are mentioned without proof in [9]. We present these here with all details since the results are not well known. In fact, these results generalize the corresponding fixed point theorems due to Kada-Suzuki-Takahashi (1996), Bae (2003), Suzuki (2005), Khamsi (2008) and others.

## 2. Preliminaries

Let $X$ be a metric space with metric $d$. We use $2^X$ to denote the collection of all nonempty subsets of $X$. A point $x \in X$ is called a fixed point of a map $f : X \to X$ ($T : X \to 2^X$) if $x = f(x)$ ($x \in T(x)$).

Recall that a real-valued function $\varphi$ defined on $X$ is said to be *lower (upper) semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n)$ ($\varphi(x) \geq \limsup_{n\to\infty} \varphi(x_n)$).

In 1976, Caristi [3] obtained the following fixed point theorem on complete metric spaces, known as Caristi's fixed point theorem.

**Theorem 2.1** Let $X$ be a complete metric space with metric $d$. Let $\psi : X \to [0, \infty)$ be a lower semicontinuous function and let $f : X \to X$ be a single valued map such that for any $x \in X$

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)). \tag{1}$$

Then $f$ has a fixed point.

To generalize Theorem 2.1, one may consider the weakening of one or more of the following hypotheses (i) the metric $d$; (ii) the lower semicontinuity of the real-valued function $\psi$; (iii) the inequality (1); (iv) the function $f$.

Using the Brezis-Browder order principle, Bae et al.[2] studied some generalizations of the Caristi's theorem. The main results are the following:

**Theorem 2.2** *Let $(X, d)$ be a complete metric space. Let $\psi : X \to [0, \infty)$ be a lower semicontinuous function and let $f : X \to X$ be a map such that for each $x \in X$,*

$$d(x, f(x)) \leq \max\{c(\psi(x)), c(\psi(f(x)))\}(\psi(x) - \psi(f(x))),$$

*where $c : [0, \infty) \to (0, \infty)$ is an upper semicontinuous function from the right. Then, $f$ has a fixed point in $X$*

**Theorem 2.3** *Let $(X, d)$ be a complete metric space. Let $\eta : [0, \infty) \to [0, \infty)$ be lower semicontinuous function such that $\eta(0) = 0$ and $\eta(t) > 0$ for $t > 0$ and*

$$\limsup_{t\to 0^+} \frac{t}{\eta(t)} < \infty.$$

*Let $f : X \to X$ be a map such that for each $x \in X$, $d(x, f(x)) \leq \psi(x)$ and*

$$\eta(d(x, f(x)) \leq \psi(x) - \psi(f(x)),$$

*where $\psi : X \to [0, \infty)$ is lower semicontinuous function. Then, $f$ has a fixed point in $X$.*

In fact, in Theorem 2.2 ([2, Theorem 3]) the authors only assumed that the function $c$ is upper semicontinuous. But, in the proof they used only the condition that $c$ is upper semicontinuous from the right. Applying Theorem 2.2 and Theorem 2.3, Bae [1] obtained fixed point results for weakly contractive multivalued maps.

Recently, Suzuki [11] generalized the results in [1, 2]. The main result in [11] is the following:

**Theorem 2.4** *Let $(X, d)$ be a complete metric space and let $g : X \to (0, \infty)$ be any function such that for some $r > 0$*

$$\sup\{g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r\} < \infty,$$

*where $\psi : X \to [0, \infty)$ is a lower semicontinuous function. Let $f : X \to X$ be a map such that for each $x \in X$,*

$$d(x, f(x)) \leq g(x)(\psi(x) - \psi(f(x))).$$

*Then $f$ has a fixed point in $X$.*

In [7], Kada et al. introduced a concept of $w$-distance on a metric space as follows:

A function $\omega : X \times X \to [0, \infty)$ is a *$w$-distance* on $X$ if it satisfies the following conditions for any $x,\ y,\ z \in X$:
($w_1$) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$;
($w_2$) the map $\omega(x, .) : X \to [0, \infty)$ is lower semicontinuous;
($w_3$) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Clearly, the metric $d$ is a $w$-distance on $X$. Let $(Y, \|.\|)$ be a normed space. Then the functions $\omega_1, \omega_2 : Y \times Y \to [0, \infty)$ defined by $\omega_1(x, y) = \|y\|$ and $\omega_2(x, y) = \|x\| + \|y\|$ for all $x, y \in Y$ are $w$-distances. Many other examples of $w$-distance are given in [7]. Note that, in general for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$ and not either of the implications $\omega(x, y) = 0 \Leftrightarrow x = y$ necessarily hold.

Using the concept of $w$-distance, Kada et al [7] generalized the Caristi's fixed point theorem as follows:

**Theorem 2.5** *Let $(X, d)$ be a complete metric space and $\omega$ be a $w$-distance on $X$. Let $f$ be a single valued self map on $X$ such that for every $x \in X$,*

$$\psi(f(x)) + \omega(x, f(x)) \leq \psi(x),$$

*where* $\psi : X \to [0, \infty)$ *is a lower semicontinuous function. Then, there exists* $x_o \in X$ *such that* $f(x_o) = x_o$ *and* $\omega(x_o, x_o) = 0$.

In [10], Susuki introduced the following notion of $\tau$-distance on metric space $(X, d)$.

A function $p : X \times X \to [0, \infty)$ is a $\tau$-*distance* on $X$ if it satisfies the following conditions for any $x,\ y,\ z \in X$:
($\tau_1$) $p(x, z) \leq p(x, y) + p(y, z)$;
($\tau_2$) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \geq 0$, and $\eta$ is concave and continuous in its second variable;
($\tau_3$) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(u, x) \leq \lim_n inf p(u, x_n$ for all $u \in X$;
($\tau_4$) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;
($\tau_5$) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.
It has been observed in [10] that ($\tau_2$) can be replaced with
$(\tau_2)^{/}$ $\inf\{\eta(x, t) : t \geq 0\} = 0$ for all $x \in X$, and $\eta$ is nondecreasing in its second variable.

In general, a $\tau$-distance $p$ does not necessarily satisfy $p(x, x) = 0$. The metric $d$ is a $\tau$-distance on $X$. Many examples and properties of $\tau$-distance are given in [10]. Here, we state some useful examples of $\tau$-distance (see; [10]).

**Proposition 2.1** *Let* $p$ *be a* $w$*-distance on a metric space* $(X, d)$*. Then* $p$ *is also a* $\tau$*- distance on* $X$*.*

**Proposition 2.2** *Let* $p$ *be a* $\tau$*-distance on a metric space* $(X, d)$*. Let* $f$ *be a selfmap on* $X$ *and* $q : X \times X \to [0, \infty)$ *defined by*

$$q(x, y) = \max\{p(fx, fy), p(fx, y)\},$$

*for all* $x, y \in X$ *is also a* $\tau$*-distance on* $X$*.*

**Proposition 2.3** *Let* $p$ *be a* $\tau$*-distance on a metric space* $(X, d)$ *and let* $c$ *be a positive real number. Then a function* $q : X \times X \to [0, \infty)$ *defined by* $q(x, y) = c\,p(x, y)$ *for all* $x, y \in X$ *is also a* $\tau$ *distance on* $X$*.*

In [10], Susuki improved Theorem 2.5 as follows:

**Theorem 2.6** *Let $(X, d)$ be a complete metric space and $p$ be a $\tau$-distance on $X$. Let $f$ be a single valued self map on $X$ such that for every $x \in X$,*

$$\psi(f(x)) + p(x, f(x)) \leq \psi(x),$$

*where $\psi : X \to [0, \infty)$ is a lower semicontinuous function. Then, there exists $x_o \in X$ such that $f(x_o) = x_o$ and $p(x_o, x_o) = 0$.*

Using Theorem 2.6, Suzuki [11] generalized Theorem 2.6 and Theorem 2.4 as follows.

**Theorem 2.7** *Let $(X, d)$ be a complete metric space, $p$ be a $\tau$-distance on $X$ and let $g : X \to (0, \infty)$ be a function such that for some $r > 0$*

$$\sup\{g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r\} < \infty,$$

*where $\psi : X \to (0, \infty)$ is a lower semicontinuous function. Let $f : X \to X$ be a map such that for each $x \in X$,*

$$p(x, f(x)) \leq g(x)(\psi(x) - \psi(f(x))).$$

*Then there exists $x_o \in X$ such that $f(x_o) = x_o$ and $p(x_o, x_o) = 0$.*

## 3. The Results

Applying Theorem 2.6, we prove the following multivalued Caristi type fixed point result

**Theorem 3.1** *Let $(X, d)$ be a complete metric space and let $p$ be a $\tau$-distance on $X$. Let $g : X \to (0, \infty)$ be any function such that for some $r > 0$*

$$\sup\{g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r\} < \infty,$$

*where $\psi : X \to [0, \infty)$ is a lower semicontinuous function. Let $T : X \to 2^X$ be a multivalued map such that for any $x \in X$, there exists $y \in T(x)$ satisfying*

$$p(x, y) \leq g(x)(\psi(x) - \psi(y)).$$

*Then $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.*

**Proof.** Define a function $f : X \to X$ by $f(x) = y \in T(x) \subseteq X$. Note that for each $x \in X$, we have

$$p(x, f(x)) \leq g(x)(\psi(x) - \psi(f(x))).$$

Now, since $g(x) > 0$, it follows that $\psi(f(x)) \leq \psi(x)$. Put

$$M = \{x \in X : \psi(x) \leq \inf_{z \in X} \psi(z) + r\} \qquad \text{and} \qquad \alpha = \sup_{z \in M} g(z) < \infty$$

Note that $M$ is nonempty closed subset of a complete metric space $X$ and hence it is complete. Now, we show that $f(M) \subseteq M$. Let $u \in M$ and $f(u) = v \in T(u)$ then we have

$$\psi(f(u)) \leq \psi(u) \leq \inf_{z \in X} \psi(z) + r$$

and thus $f(u) \in M$ and hence $f$ is a self map on $M$. Note that $\alpha\psi$ is lower semicontinuous and for each $x \in M$ we have

$$p(x, f(x)) \leq \alpha(\psi(x)) - \alpha(\psi(f(x))).$$

By Theorem 2.6, there exists $x_0 \in M$ such that $f(x_0) = x_0 \in T(x_0)$. and $p(x_0, x_0) = 0$.

Now, using Theorem 3.1, we prove multivalued generalized Caristi's fixed point results with respect to $\tau$-distance.

**Theorem 3.2** *Let $(X, d)$ be a complete metric space and let $p$ be a $\tau$-distance on $X$. Let $T : X \to 2^X$ be a multivalued map such that for any $x \in X$, there exists $y \in T(x)$ satisfying*

$$p(x, y) \leq \max\{c(\psi(x)), c(\psi(y))\}(\psi(x) - \psi(y)),$$

*where $\psi : X \to [0, \infty)$ is a lower semicontinuous function and $c : [0, \infty) \to (0, \infty)$ is an upper semicontinuous from the right. Then $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.*

**Proof.** Put $t_0 = \inf_{x \in X} \psi(x)$. By the definition of the function $c$, there exist some positive real numbers $r, r_0$ such that $c(t) \leq r_0$ for all $t \in [t_0, t_0 + r]$. Now, for all $x \in X$ we define

$$g(x) = \max\{c(\psi(x)), c(\psi(y))\}.$$

Clearly, $g$ maps $X$ into $(0, \infty)$. Note that for all $x \in X$, we get $\psi(y) \leq \psi(x)$ and thus for any $x \in X$ with $\psi(x) \leq t_0 + r$, we have

$$\psi(y) \leq t_0 + r.$$

Now, clearly, $g(x) \leq r_0 < \infty$ and hence we obtain

$$\sup\{g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r\} < \infty$$

By Theorem 3.1, $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

**Theorem 3.3** *Let $(X, d)$ be a complete metric space and let $p$ be a $\tau$-distance on $X$. Let $T : X \to 2^X$ be a multivalued map such that for any $x \in X$, there exists $y \in T(x)$ satisfying*

$$p(x, y) \leq c(\psi(x))(\psi(x) - \psi(y)),$$

*where $\psi : X \to [0, \infty)$ is a lower semicontinuous function and $c : [0, \infty) \to (0, \infty)$ is a nondecreasing function. Then $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.*

**Proof.** For each $x \in X$, define $g(x) = c(\psi(x))$. Clearly, $g$ does carry $X$ into $(0, \infty)$. Now, since the function $c$ is nondecreasing, for any real number $r > 0$ we have

$$\sup\{g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r\} \leq c(\inf_{z \in X} \psi(z) + r) < \infty.$$

Thus, by Theorem 3.1, the result follows.

**Corollary 3.4** *Let $(X, d)$ be a complete metric space and let $p$ be a $\tau$-distance on $X$. Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$p(x, y) \leq c(\psi(y))(\psi(x) - \psi(y)),$$

*where $\psi : X \to [0, \infty)$ is a lower semicontinuous function and $c : [0, \infty) \to (0, \infty)$ is a nondecreasing function. Then $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.*

**Proof.** Since for each $x \in X$ there is $y \in T(x)$ such that $\psi(y) \leq \psi(x)$ and the function $c$ is nondecreasing, we have $c(\psi(y)) \leq c(\psi(x))$. Thus the result follows from Theorem 3.3.

Applying Theorem 3.3, we prove the following fixed point result.

**Theorem 3.5** *Let $(X, d)$ be a complete metric space and let $p$ be a $\tau$-distance on $X$. Let $T : X \to 2^X$ be a multivalued map such that for any $x \in X$, there exists $y \in T(x)$ satisfying $p(x, y) \leq \psi(x)$ and*

$$p(x, y) \leq \eta(p(x, y))(\psi(x) - \psi(y)),$$

*where $\psi : X \to [0, \infty)$ is a lower semicontinuous function and $\eta : [0, \infty) \to (0, \infty)$ is upper semicontinuous function. Then $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.*

**Proof.** Define a function $c$ from $[0, \infty)$ into $(0, \infty)$ by

$$c(t) = \sup\{\eta(r) : 0 \le r \le t\}.$$

Clearly, $c$ is nondecreasing function. Now, since $p(x, y) \le \psi(x)$, we have $c(p(x, y)) \le c(\psi(x))$. Thus by Theorem 3.3, the result follows.

The following result can be seen as a generalization of [8, Theorem 4] and [5, Theorem 4.2].

**Corollary 3.6** *Let $(X, d)$ be a complete metric space and let $p$ be a $\tau$-distance on $X$. Let $\phi : [0, \infty) \to [0, \infty)$ be lower semicontinuous function such that*

$$\limsup_{t \to 0^+} \frac{t}{\phi(t)} < \infty.$$

*Let $T : X \to 2^X$ be a multivalued map such that for any $x \in X$, there exists $y \in T(x)$ satisfying $p(x, y) \le \psi(x)$ and*

$$\phi(p(x, y) \le \psi(x) - \psi(y).$$

*Then $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.*

**Proof.** Define a function $\eta : [0, \infty) \to (0, \infty)$ by

$$\eta(0) = \limsup_{t \to 0^+} \frac{t}{\phi(t)} \qquad \text{and} \qquad \eta(t) = \frac{t}{\phi(t)},\ t > 0.$$

Then $\eta$ is upper semicontinuous. Also note that

$$p(x, y) \le \eta(p(x, y))(\psi(x) - \psi(y)).$$

Thus by Theorem 3.5, $T$ has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

**Remark 3.7** a) Theorem 3.1 is a multivalued version of the generalized Caristi's fixed point Theorem 3.1 ([11, Theorem 7]) and is also a generalization of Theorem 2.4 ([11, Theorem 2]) and Theorem 2.5 ([7, Theorem 2]).

b) Theorem 3.2 is a multivalued version of the result [11, Theorem 8]) and contains Theorem 2.2 ( [2, Theorem 3]) as a special case.

c) Theorem 3.3 is a multivalued version of [11, Theorem 9]) and contains [1, Theorem 2.3] as a special case.

d) Theorem 3.5 is a multivalued version of [11, Theorem 10]) and contains [1, Theorem 2.4] as a special case.

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