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SOME RESULTS CONCERNING CALCULATION OF THE TEST FUNCTIONS BY BERNSTEIN TYPE OPERATORS

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Abstract. In the present paper, we establish some results concerning calculation of the test functions by Bernstein type operators and also present in every case an appropriate application.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
The operators $B_n : C[0,1] \to C[0,1]$ given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where $p_{n,k}$ are the fundamental Bernstein's polynomials defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{2}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n\}$ and any $n \in \mathbb{N}$, are called Bernstein operators. These operators were first introduced by S.N. Bernstein [5] in 1912.
Let $p \in \mathbb{N}_0$ be a fixed natural number. The operators $\tilde{B}_{n,p} : C[0,1+p] \to C[0,1]$ given by

$$\tilde{B}_{n,p}(f;x) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}(x) f\left(\frac{k}{n}\right), \tag{3}$$

where $\tilde{p}_{n,k}$ are the fundamental Schurer's polynomials defined by

$$\tilde{p}_{n,k}(x) = \binom{n+p}{k} x^k (1-x)^{n+p-k}, \tag{4}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n+p\}$ and any $n \in \mathbb{N}$, are called Schurer operators [8].
Let the real parameters α, β be given, such that $0 \le \alpha \le \beta$. The operators $P_n^{(\alpha,\beta)}$: $C[0,1] \to C[0,1]$ defined by

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \tag{5}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n\}$ and any $n \in \mathbb{N}$, where $p_{n,k}(x)$ are the fundamental Bernstein's polynomials given at (2), are called Stancu operators [9].
Let $p \in \mathbb{N}_0$ be a fixed natural number and let the real parameters α, β be given, such that $0 \le \alpha \le \beta$. The operators $\tilde{S}_{n,p}^{(\alpha,\beta)} : C[0,1+p] \to C[0,1]$ defined by

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \tag{6}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n+p\}$ and any $n \in \mathbb{N}$, where $\tilde{p}_{n,k}(x)$ are the fundamental Schurer's polynomials given at (4), are called Schurer-Stancu operators. These operators were first introduced by H.H. Gonska and J. Meier [6], then studied intensively by D. Bărbosu [3], [4].

Remark 1. More results and properties concerning (1), (3), (5) and (6) can be found also in monographs [2], [1], [4].

The purpose of this paper is to establish in every case a general result concerning calculation of the test functions by Bernstein type operators, similar with the result proved by O.T. Pop, D. Bărbosu and P.I. Braica [7], for Bernstein operators.

2. Preliminaries

In what follows, we recall the main results from [7], which we shall use afterwards in the paper. Before to mention the results we set $\binom{n}{k} = 0$ and $A_n^k = 0$, where A_n^k are arrangements of n taken k, for any $n \in \mathbb{N}_0$ and $k \in \mathbb{Z} \setminus \{0,1,\ldots,n\}$.
Let $e_j(x) = x^j$, with $j \in \mathbb{N}_0$ be the test functions.

Lemma 1. [7] *For any $k, n \in \mathbb{N}_0$ and $j \in \mathbb{N}$, the following identity*

$$k^j \binom{n}{k} = \sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i} \binom{n-j+i}{k-j+i} \tag{7}$$

holds, where

$$a_j^{(i)} > 0, \quad i = \overline{1, j-2}, \qquad a_j^{(0)} = a_j^{(j-1)} = 1 \tag{8}$$

and

$$a_{j+1}^{(i)} = (j-i+1)a_j^{(i-1)} + a_j^{(i)},\ \textit{for } 1 \le i \le j-1. \tag{9}$$

Theorem 1. [7] *For any $j, n \in \mathbb{N}$ and any $x \in [0,1]$, the following*

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i} x^{j-i} \tag{10}$$

holds.

3. Main results

We set $\binom{n+p}{k} = 0$ and $A_{n+p}^k = 0$, for any $n, p \in \mathbb{N}_0$, with $p \neq 0$ and $k \in \mathbb{Z}\backslash\{0,1,\ldots,n+p\}$.

Lemma 2. *For any $k, n, p \in \mathbb{N}_0$, with $p \neq 0$, the following identity*

$$k^j \binom{n+p}{k} = \sum_{i=0}^{j-1} a_j^{(i)} A_{n+p}^{j-i} \binom{n+p-j+i}{k-j+i} \tag{11}$$

holds, where the coefficients $a_j^{(i)}$ are given at (8) *and* (9).

Proof. We assume that (11) holds and taking into account the mathematical induction with respect j, we get

$$\begin{aligned} k^{j+1}\binom{n+p}{k} &= k \cdot k^j \binom{n+p}{k} = k\sum_{i=0}^{j-1} a_j^{(i)} A_{n+p}^{j-i}\binom{n+p-j+i}{k-j+i} \\ &= \sum_{i=0}^{j-1}((k-j+i)+(j-i))a_j^{(i)} A_{n+p}^{j-i}\binom{n+p-j+i}{k-j+i} \\ &= a_j^{(0)} A_{n+p}^{j+1}\binom{n+p-j-1}{k-j-1} + \left(ja_j^{(0)} + a_j^{(1)}\right) A_{n+p}^{j}\binom{n+p-j}{k-j} + \ldots \\ &\quad + \left(2a_j^{(j-2)} + a_j^{(j-1)}\right) A_{n+p}^2 \binom{n+p-2}{k-2} + a_j^{(j-1)} A_{n+p}^1 \binom{n+p-1}{k-1}. \end{aligned}$$

Taking (8) and (9) into account, it follows

$$k^{j+1}\binom{n+p}{k} = \sum_{i=0}^{j} a_{j+1}^{(i)} A_{n+p}^{j-i+i}\binom{n+p-j+i-1}{k-j+i-1}.$$

In the case of Schurer operators, we get:

Theorem 2. *For any $j, n, p \in \mathbb{N}$ and any $x \in [0,1]$, the following*

$$\tilde{B}_{n,p}(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1} a_j^{(i)} A_{n+p}^{j-i} x^{j-i} \tag{12}$$

holds.

Proof.

$$\tilde{B}_{n,p}(e_j;x) = \sum_{k=0}^{n+p}\binom{n+p}{k}x^k(1-x)^{n+p-k}\left(\frac{k}{n}\right)^j = \frac{1}{n^j}\sum_{k=0}^{n+p}x^k(1-x)^{n+p-k}k^j\binom{n+p}{k}.$$

Taking (11) into account, it follows

$$\begin{aligned}\tilde{B}_{n,p}(e_j;x) &= \frac{1}{n^j}\sum_{k=0}^{n+p}x^k(1-x)^{n+p-k}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}\binom{n+p-j+i}{k-j+i}\\ &= \frac{1}{n^j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}\sum_{k=0}^{n+p}\binom{n+p-j+i}{k-j+i}x^{k-j+i}(1-x)^{n+p-k}\\ &= \frac{1}{n^j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}.\end{aligned}$$

Application 1. For $j \in \{1,2,3,4\}$, we present the first four cases concerning calculation of the test functions by Schurer operators.

Case 1. $j = 1$

$$\tilde{B}_{n,p}(e_1;x) = \frac{1}{n}a_1^{(0)}A_{n+p}^1 x = \frac{(n+p)x}{n}.$$

Case 2. $j = 2$

$$\begin{aligned}\tilde{B}_{n,p}(e_2;x) &= \frac{1}{n^2}\sum_{i=0}^{1}a_2^{(i)}A_{n+p}^{2-i}x^{2-i} = \frac{1}{n^2}\left(a_2^{(0)}A_{n+p}^2x^2 + a_2^{(1)}A_{n+p}^1x\right)\\ &= \frac{(n+p)(n+p-1)x^2}{n^2} + \frac{(n+p)x}{n^2}.\end{aligned}$$

Case 3. $j = 3$

$$\begin{aligned}\tilde{B}_{n,p}(e_3;x) &= \frac{1}{n^3}\sum_{i=0}^{2}a_3^{(i)}A_{n+p}^{3-i}x^{3-i} = \frac{1}{n^3}\left(a_3^{(0)}A_{n+p}^3x^3 + a_3^{(1)}A_{n+p}^2x^2 + a_3^{(2)}A_{n+p}^1x\right)\\ &= \frac{A_{n+p}^3x^3}{n^3} + \frac{3A_{n+p}^2x^2}{n^3} + \frac{A_{n+p}^1x}{n^3},\end{aligned}$$

where $a_3^{(1)} = 2a_2^{(0)} + a_2^{(1)} = 3$.

Case 4. $j = 4$

$$\tilde{B}_{n,p}(e_4;x) = \frac{1}{n^4}\sum_{i=0}^{3} a_4^{(i)} A_{n+p}^{4-i} x^{4-i}$$

$$= \frac{1}{n^4}\left(a_4^{(0)}A_{n+p}^4 x^4 + a_4^{(1)}A_{n+p}^3 x^3 + a_4^{(2)}A_{n+p}^2 x^2 + a_4^{(3)}A_{n+p}^1 x\right)$$

$$= \frac{A_{n+p}^4 x^4}{n^4} + \frac{6A_{n+p}^3 x^3}{n^4} + \frac{7A_{n+p}^2 x^2}{n^4} + \frac{A_{n+p}^1 x}{n^4},$$

where $a_4^{(1)} = 3a_3^{(0)} + a_3^{(1)} = 6$, $a_4^{(2)} = 2a_3^{(1)} + a_3^{(2)} = 7$.

In the case of Stancu operators, we get:

Theorem 3. *For any $l, n \in \mathbb{N}$, any $x \in [0,1]$ and $0 \le \alpha \le \beta$, the following*

$$P_n^{(\alpha,\beta)}(e_l;x) = \frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i} x^{j-i} \tag{13}$$

holds.

Proof.

$$P_n^{(\alpha,\beta)}(e_l;x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left(\frac{k+\alpha}{n+\beta}\right)^l = \frac{1}{(n+\beta)^l}\sum_{k=0}^{n} x^k(1-x)^{n-k}(k+\alpha)^l\binom{n}{k}$$

$$= \frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{k=0}^{n} x^k(1-x)^{n-k}k^j\binom{n}{k}.$$

Taking (7) into account, it follows

$$P_n^{(\alpha,\beta)}(e_l;x) = \frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{k=0}^{n} x^k(1-x)^{n-k}\sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i}\binom{n-j+i}{k-j+i}$$

$$= \frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i} x^{j-i}\sum_{k=0}^{n}\binom{n-j+i}{k-j+i}x^{k-j+i}(1-x)^{n-k}$$

$$= \frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i} x^{j-i}.$$

Application 2. For $l \in \{1,2,3\}$, we present the first three cases concerning calculation of the test functions by Stancu operators.

Case 1. l=1

$$P_n^{(\alpha,\beta)}(e_1;x)=\frac{1}{n+\beta}\sum_{j=0}^{1}\binom{1}{j}\alpha^{1-j}\sum_{i=0}^{j-1}a_j^{(i)}A_n^{j-i}x^{j-i}=\frac{1}{n+\beta}\left(\alpha+a_1^{(0)}A_n^1x\right)=\frac{\alpha+nx}{n+\beta}.$$

Case 2. l=2

$$\begin{aligned}P_n^{(\alpha,\beta)}(e_2;x)&=\frac{1}{(n+\beta)^2}\sum_{j=0}^{2}\binom{2}{j}\alpha^{2-j}\sum_{i=0}^{j-1}a_j^{(i)}A_n^{j-i}x^{j-i}\\&=\frac{1}{(n+\beta)^2}\left(\alpha^2+2\alpha a_1^{(0)}A_n^1x+a_2^{(0)}A_n^2x^2+a_2^{(1)}A_n^1x\right)\\&=\frac{1}{(n+\beta)^2}\left(\alpha^2+2\alpha nx+n(n-1)x^2+nx\right)\\&=\frac{n(n-1)x^2}{(n+\beta)^2}+\frac{(2\alpha+1)nx}{(n+\beta)^2}+\frac{\alpha^2}{(n+\beta)^2}.\end{aligned}$$

Case 3. l=3

$$P_n^{(\alpha,\beta)}(e_3;x)=\frac{1}{(n+\beta)^3}\sum_{j=0}^{3}\binom{3}{j}\alpha^{3-j}\sum_{i=0}^{j-1}a_j^{(i)}A_n^{j-i}x^{j-i}=\frac{1}{(n+\beta)^3}\left(\alpha^3\right.$$

$$\left.+3\alpha^2a_1^{(0)}A_n^1x+3\alpha\left(a_2^{(0)}A_n^2x^2+a_2^{(1)}A_n^1x\right)+a_3^{(0)}A_n^3x^3+a_3^{(1)}A_n^2x^2+a_3^{(2)}A_n^1x\right)$$

$$=\frac{n(n-1)(n-2)x^3}{(n+\beta)^3}+\frac{3(\alpha+1)n(n-1)x^2}{(n+\beta)^3}+\frac{(3\alpha^2+3\alpha+1)nx}{(n+\beta)^3}+\frac{\alpha^3}{(n+\beta)^3}.$$

In the case of Schurer-Stancu operators, we get:

Theorem 4. *For any $l,n,p\in\mathbb{N}$, any $x\in[0,1]$ and $0\le\alpha\le\beta$, the following*

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(e_l;x)=\frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}\tag{14}$$

holds.

Proof.

$$\begin{aligned}\tilde{S}_{n,p}^{(\alpha,\beta)}(e_l;x)&=\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)\left(\frac{k+\alpha}{n+\beta}\right)^l\\&=\frac{1}{(n+\beta)^l}\sum_{k=0}^{n+p}x^k(1-x)^{n+p-k}(k+\alpha)^l\binom{n+p}{k}\\&=\frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{k=0}^{n+p}x^k(1-x)^{n+p-k}k^j\binom{n+p}{k}.\end{aligned}$$

Taking (11) into account, it follows

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(e_l;x)=\frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{k=0}^{n+p}x^k(1-x)^{n+p-k}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}\binom{n+p-j+i}{k-j+i}$$

$$=\frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}\sum_{k=0}^{n+p}\binom{n+p-j+i}{k-j+i}x^{k-j+i}(1-x)^{n+p-k}$$

$$=\frac{1}{(n+\beta)^l}\sum_{j=0}^{l}\binom{l}{j}\alpha^{l-j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}.$$

Application 4. For $l\in\{1,2,3\}$, we present the first three cases concerning calculation of the test functions by Schurer-Stancu operators.

Case 1. $l=1$

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(e_1;x)=\frac{1}{n+\beta}\sum_{j=0}^{1}\binom{1}{j}\alpha^{1-j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}$$

$$=\frac{1}{n+\beta}\left(\alpha+a_1^{(0)}A_{n+p}^{1}x\right)=\frac{(n+p)x}{n+\beta}+\frac{\alpha}{n+\beta}.$$

Case 2. $l=2$

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(e_2;x)=\frac{1}{(n+\beta)^2}\sum_{j=0}^{2}\binom{2}{j}\alpha^{2-j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}$$

$$=\frac{1}{(n+\beta)^2}\left(\alpha^2+2\alpha a_1^{(0)}A_{n+p}^{1}x+a_2^{(0)}A_{n+p}^{2}x^2+a_2^{(1)}A_{n+p}^{1}x\right)$$

$$=\frac{A_{n+p}^{2}x^2}{(n+\beta)^2}+\frac{(2\alpha+1)A_{n+p}^{1}x}{(n+\beta)^2}+\frac{\alpha^2}{(n+\beta)^2}.$$

Case 3. $l=3$

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(e_3;x)=\frac{1}{(n+\beta)^3}\sum_{j=0}^{3}\binom{3}{j}\alpha^{3-j}\sum_{i=0}^{j-1}a_j^{(i)}A_{n+p}^{j-i}x^{j-i}$$

$$=\frac{1}{(n+\beta)^3}\left(\alpha^3+\alpha^2a_1^{(0)}A_{n+p}^{1}x+\alpha\left(a_2^{(0)}A_{n+p}^{2}x^2+a_2^{(1)}A_{n+p}^{1}x\right)\right.$$

$$\left.+a_3^{(0)}A_{n+p}^{3}x^3+a_3^{(1)}A_{n+p}^{2}x^2+a_3^{(2)}A_{n+p}^{1}x\right)$$

$$=\frac{A_{n+p}^{3}x^3}{(n+\beta)^3}+\frac{(\alpha+3)A_{n+p}^{2}x^2}{(n+\beta)^3}+\frac{\left(\alpha^2+\alpha+1\right)A_{n+p}^{1}x}{(n+\beta)^3}+\frac{\alpha^3}{(n+\beta)^3}.$$

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