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APPLICATIONS OF A COMBINED MONTE CARLO AND QUASI-MONTE CARLO METHOD TO PRICING BARRIER OPTIONS

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Abstract. In this paper, we apply a combined Monte Carlo and Quasi-Monte Carlo method, developed by us in a previous paper [31], to the evaluation of barrier options. We assume that the stock price of the underlying asset is driven by a Lévy process with independent increments distributed according to a NIG distribution. We also provide numerical results that compare our method with the Monte Carlo method. Numerical experiments indicate an increased accuracy of our method.

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1. Introduction

Barrier options are one of the most important derivatives in the financial markets. In the case of barrier options the general idea is that the payoff depends on whether the underlying asset price hits a predetermined barrier level ([16]). In this paper we evaluate by simulation the Up-and-Out barrier options and Double Knock-Out barrier options, in the situation where the stock price is modeled by an exponential Lévy process. For the Up-and-Out barrier option, the option is valid only as long as the upper barrier is never touched during the life of the option. For the Double Knock-Out barrier options the option is valid only as long as the underlying asset remains above the lower barrier and bellow the upper barrier until maturity. If the asset price touches either the upper or the lower barrier, then the option is knocked out worthless (zero payoff).

Simulation techniques such as Monte Carlo (MC) and Quasi-Monte Carlo (QMC) methods play a key role in the evaluation of options with assets having non-normal increments because of the difficulty in obtaining general analytical solutions. Boyle [2] used MC simulation and diverse variance reduction techniques to estimate the value of barrier options in the Black-Scholes modeling framework of a financial market. Applications of the QMC method to option pricing problems can be found in [7], [13] and [17].

Barndorff-Nielsen [1] proposed to model the log-returns, by using the normal inverse Gaussian (NIG) distribution, as this class of distributions has proven to fit the semi-heavily tails observed in financial time series of various kinds extremely well [6], [33]. A method for evaluating such derivatives is the one proposed by Raible [26], who considered a Fourier method to evaluate call and put options. Other alternative methods for evaluating such derivatives are the MC and QMC methods. In [14], Kainhofer proposes a QMC algorithm for generating NIG variables, based on a technique proposed by Hlawka and Mück [11, 12] for generating low-discrepancy sequences for general distributions.

In an earlier paper [31], we developed a combined MC and QMC method, to estimate a multidimensional integral I of a function f, with respect to the probability measure induced by a distribution function G on $[0,1]^s$. Our method is based on random sampling from sequences with low G-discrepancy. Other methods that combine the ideas of MC and QMC methods and their applications to option pricing can be found in [19], [20], [22], [27], [8], [28] .

In this paper, we first recall the general setting of our combined method and give some theoretical results. Next, we apply our method to the evaluation of an Up-and-Out barrier option and of a Double Knock-Out barrier option. We assume that the stock price of the underlying asset $S = S(t)$ is driven by a Lévy process $Z(t)$, with independent increments distributed according to a NIG distribution. We compare the estimate produced by our method with the estimate given by MC method.

2. Monte Carlo and Quasi-Monte Carlo methods

We consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function G and density function g (g is nonnegative and $\int_{[0,1]^s} g(u)du = 1$).

We consider the problem of approximating the multidimensional integral of a function $f : [0,1]^s \to \mathbb{R}$, of the form

$$I = \int_{[0,1]^s} f(x)dG(x) = \int_{[0,1]^s} f(x)g(x)dx. \quad (1)$$

Two frequently used approaches are the MC and QMC methods.

In the MC method, we generate N independent sample variables $X_1, \ldots, X_N$, with the density function g on $[0,1]^s$. The integral I is estimated by the sample mean

$$\bar{I}_{MC} = \frac{1}{N}\sum_{k=1}^{N} f(X_k).$$

The estimator $\bar{I}_{MC}$ is an unbiased estimator of the integral I. The strong law of large numbers tells us that

$$P\Big(\lim_{N\to\infty} \bar{I}_{MC} = I\Big) = 1.$$

In other words, the MC estimator converges almost surely to I, as $N \to \infty$.

By the central limit theorem, the MC method provides probabilistic error bounds of order $O\big(1/\sqrt{N}\big)$.

The QMC method can be defined by analogy with the MC method, by replacing the random samples by a sequence of "well distributed" deterministic points. This approach uses the so-called *sequences with low G-discrepancy* in $[0,1]^s$. We define these sequences, using the notions of *G-star discrepancy* and *G-discrepancy*.

Definition 1 (G-star discrepancy). *We consider a distribution on $[0,1]^s$, with distribution function G. Let λ_G be the probability measure induced by G. Let $P = (x_1, \dots, x_N)$ be a set of points in $[0,1]^s$. The G-star discrepancy of P is defined as*

$$D^*_{N,G}(P) = D^*_{N,G}(x_1,\dots,x_N) = \sup_{J\subseteq[0,1]^s} \left|\frac{1}{N}A_N(J,P) - \lambda_G(J)\right|,$$

where the supremum is calculated over all subintervals J of $[0,1]^s$ of the form $\prod_{i=1}^{s}[0,a_i]$, and $A_N(J,P)$ counts the number of elements of P falling into the interval J, i.e.,

$$A_N(J,P) = \sum_{k=1}^{N} 1_J(x_k),$$

where 1_J is the characteristic function of J.

Definition 2 (G-discrepancy). *Under the same conditions as in Definition 1, the G-discrepancy of $P = (x_1, \dots, x_N)$ is defined as*

$$D_{N,G}(P) = D_{N,G}(x_1,\dots,x_N) = \sup_{J\subseteq[0,1]^s} \left|\frac{1}{N}A_N(J,P) - \lambda_G(J)\right|,$$

where the supremum is calculated over all subintervals J of $[0,1]^s$ of the form $\prod_{i=1}^{s}[a_i,b_i]$.

The notions of G-star discrepancy and G-discrepancy are natural generalizations of the notions of star discrepancy and discrepancy, respectively, which are used in the uniform case [18].

For a sequence $P = (x_k)_{k\geq 1}$ of points in $[0,1]^s$, we write $D^*_{N,G}(P)$ for the G-star discrepancy and $D_{N,G}(P)$ for the G-discrepancy of the first N terms of sequence P.

Definition 3 (sequence of points with low G-discrepancy). *A sequence of points $P = (x_k)_{k\geq 1}$, with $x_k \in [0,1]^s$, $k \geq 1$, is said to be with low G-discrepancy if we have*

$$D_{N,G}(P) = O\left(\frac{(\log N)^s}{N}\right) \quad \textit{for all } N \geq 2.$$

Sequences with low G-discrepancy are used in QMC integration to approximate the integral (1). Several methods for generating such sequences are proposed in [9], [10], [29] and [30].

The QMC integration formula is

$$I = \int_{[0,1]^s} f(x)dG(x) \approx \frac{1}{N}\sum_{k=1}^{N} f(x_k), \tag{2}$$

where $(x_k)_{k\geq 1}$ is a sequence with low G-discrepancy in $[0,1]^s$.

The non-uniform Koksma-Hlawka inequality gives an upper bound for the error of approximation in formula (2).

Theorem 4 (non-uniform Koksma-Hlawka inequality). *([3], [21]) Let $f : [0,1]^s \to \mathbb{R}$ be a function of bounded variation in the sense of Hardy and Krause. We consider a distribution on $[0,1]^s$, with distribution function G. Then, for any $x_1, \ldots, x_N \in [0,1]^s$, we have*

$$\left|\frac{1}{N}\sum_{k=1}^{N} f(x_k) - \int_{[0,1]^s} f(x)dG(x)\right| \leq V_{HK}(f) D^*_{N,G}(x_1, \ldots, x_N). \tag{3}$$

In [31] (see also [32]), we proposed a combined MC and QMC method based on random sampling from sequences with low G-discrepancy in $[0,1]^s$. Next, we describe our method.

3. Estimation of integrals using random sampling from sequences with low G-discrepancy in $[0,1]^s$

Our combined MC and QMC method for estimating the multidimensional integral I, given by (1), consists of the following.

We consider a distribution on $[0,1]^s$, with distribution function G and density function g. We use the marginal density functions g_l, $l = 1, \ldots, s$, and the marginal distribution functions G_l, $l = 1, \ldots, s$, defined as follows.

Definition 5. *Consider a distribution on* $[0,1]^s$*, with density function* g*. For a point* $u = \left(u^{(1)}, \dots, u^{(s)}\right) \in [0,1]^s$*, the marginal density functions* g_l*,* $l = 1, \dots, s$*, are defined by*

$$g_l\left(u^{(l)}\right) = \underbrace{\int \dots \int}_{[0,1]^{s-1}} g\left(t^{(1)}, \dots, t^{(l-1)}, u^{(l)}, t^{(l+1)}, \dots t^{(s)}\right) dt^{(1)} \dots dt^{(l-1)} dt^{(l+1)} \dots dt^{(s)},$$

and the marginal distribution functions G_l*,* $l = 1, \dots, s$*, are defined by*

$$G_l\left(u^{(l)}\right) = \int_0^{u^{(l)}} g_l(t)dt.$$

We assume that $G(u) = \prod_{l=1}^{s} G_l(u^{(l)})$, $\forall u = \left(u^{(1)}, \dots, u^{(s)}\right) \in [0,1]^s$ (independent marginals). Moreover, we assume that the functions G_l, $l = 1, \dots, s$, are invertible on $[0,1]$.

Let $\Omega = \{\beta_1, \dots, \beta_r\}$ be a space containing sets of points β_i, $i = 1, \dots, r$, with low G-discrepancy in $[0,1]^s$, where the point set β_i, $i = 1, \dots, r$, is of the form

$$\beta_i = (\beta_{1,i}, \dots, \beta_{N,i}),$$

with $\beta_{k,i} = (\beta_{k,i}^{(1)}, \dots, \beta_{k,i}^{(s)}) \in [0,1]^s$, $k = 1, \dots, N$.

We define the random variable X_N on the space Ω as follows.

Definition 6. *([31]) For an arbitrary point set* $\beta_i = (\beta_{1,i}, \dots, \beta_{N,i}) \in \Omega$*, the value of the random variable* X_N *is defined as*

$$X_N(\beta_i) = \frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i}),$$

and is taken with probability $1/r$*.*

Remark 7. *([31]) The distribution of the random variable* X_N *is*

$$X_N : \begin{pmatrix} \frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i}) \\ 1/r \end{pmatrix}_{\substack{\beta_i = (\beta_{1,i}, \dots, \beta_{N,i}) \\ i=1,\dots,r}}.$$

Theorem 8. *([31]) The random variable* X_N *has the following properties:*

$$\lim_{N \to \infty} E(X_N) = I, \tag{4}$$

$$\lim_{N \to \infty} Var(X_N) = 0. \tag{5}$$

Once we have defined the random variable X_N, we select the integers $i_1, \dots, i_M$ at random from the uniform distribution on $\{1, \dots, r\}$, and consider the corresponding point sets $\beta_{i_1}, \dots, \beta_{i_M}$. For each point set, we compute the value of the random variable X_N. The values $X_N(\beta_{i_1}), \dots, X_N(\beta_{i_M})$ are values of the sample variables $X_{N,i_1}, \dots, X_{N,i_M}$ that are independent identically distributed random variables and have the same distribution as X_N.

We use the notation $\overline{X}_{N,M}$ for the sample mean of the random variables X_{N,i_1}, ..., X_{N,i_M}, and $\overline{x}_{N,M}$ for its value, i.e.,

$$\begin{aligned} \overline{X}_{N,M} &= \frac{X_{N,i_1} + \ldots + X_{N,i_M}}{M}, \\ \overline{x}_{N,M} &= \frac{\sum_{l=1}^{M} X_{N,i_l}(\beta_{i_l})}{M} = \frac{\sum_{l=1}^{M} \left(\frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i_l}) \right)}{M}. \end{aligned}$$

Proposition 9. *([31]) For a fixed N, the estimator $\overline{X}_{N,M}$ has the following properties:*

$$\begin{aligned} E(\overline{X}_{N,M}) &= E(X_N), \quad (\textit{unbiased estimator of } E(X_N)), && (6) \\ Var(\overline{X}_{N,M}) &= \frac{Var(X_N)}{M}, && (7) \\ \lim_{M\to\infty} Var(\overline{X}_{N,M}) &= 0, && (8) \\ P\Big(\lim_{M\to\infty} \overline{X}_{N,M} = E(X_N) \Big) &= 1, \quad (\overline{X}_{N,M} \textit{ converges almost surely to } E(X_N)). && (9) \end{aligned}$$

Proposition 10. *([31]) For a fixed M, we have the following properties of the estimator $\overline{X}_{N,M}$:*

$$\begin{aligned} \lim_{N\to\infty} E(\overline{X}_{N,M}) &= I, \\ \lim_{N\to\infty} Var(\overline{X}_{N,M}) &= 0. \end{aligned}$$

Taking into account these properties, in our combined method the integral I is approximated by

$$I \approx \overline{x}_{N,M} = \frac{\sum_{l=1}^{M} X_{N,i_l}(\beta_{i_l})}{M} = \frac{\sum_{l=1}^{M} \left(\frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i_l}) \right)}{M}. \tag{10}$$

Hence, in our method we take a random sampling from a finite set of QMC approximations, and we consider the sample mean of that sample as an estimator for the integral I. Our combined method involves random sampling from sequences with low G-discrepancy in $[0,1]^s$ (**r**andom **s**ampling from **n**on-**u**niform sequences with low G-discrepancy). It constructs the estimator $\overline{X}_{N,M}$, which we call an RSNU estimator. We call the value $\overline{x}_{N,M}$ an RSNU estimate.

Theorem 11. *([31]) The error of approximation in the combined method is bounded by*

$$\left|I - \overline{x}_{N,M}\right| \le \frac{1}{M} V_{HK}(f) \sum_{l=1}^{M} D^*_{N,G}(\beta_{i_l}).$$

Corollary 12. *([31]) For a fixed M, the RSNU estimate satisfies the following property:*

$$\lim_{N\to\infty} \overline{x}_{N,M} = I.$$

4. Application to finance: Evaluation of Barrier Options

In the following, we apply the MC method and our combined method to pricing barrier options. We consider Up-and-Out barrier options and Double Knock-Out barrier options. Next, we present the general setting and the modeling of the problem.

We consider the situation where the stock price of the underlying asset $S = S(t)$ is driven by a Lévy process $Z(t)$,

$$S(t) = S(0)e^{Z(t)}. \tag{11}$$

Lévy processes can be characterized by the distribution of the random variable $Z(1)$. This distribution can be hyperbolic (see [6]), normal inverse gaussian (NIG), variance-gamma (see [15]), or Meixner distribution.

According to the fundamental theory of asset pricing (see [5]), the risk-neutral price of a barrier option, $C(0)$, is given by

$$C(0) = E^Q(C(\tau, S_\tau)), \tag{12}$$

where $C(\tau, S_\tau)$ is the discounted payoff of the derivative, τ is the first hitting time of the considered barrier price by the underlying asset $S(t)$ and Q is an equivalent martingale measure or a risk-neutral measure. In this paper, we are concerned with exponential NIG-Lévy processes, meaning that $Z(t)$ has independent increments, distributed according to a NIG distribution. For a detailed discussion of the NIG distribution and the corresponding Lévy process, we refer to Barndorff-Nielsen [1] and Rydberg [33]. In the situation of exponential NIG-Lévy models, we have an incomplete market, leading to a continuum of equivalent martingale measures Q, which can be used for risk-neutral pricing. Here, we choose the approach of Raible [26] and consider the measure obtained by Esscher transform method. This approach is so-called structure preserving, in the sense that the distribution of $Z(1)$ remains in the class of NIG distributions.

We consider the evaluation of Up-and-Out barrier call options, which have to be valued by simulation. For the Up-and-Out barrier option, the option is valid only as long as an upper barrier is never touched during the life of the option. The random variable τ is defined as

$$\tau = \inf\{v \geq 0 | S(v) \geq L\}, \tag{13}$$

where L is the upper barrier price. The discounted payoff of such an option is

$$C(\tau, S_\tau) = \begin{cases} e^{-rT}(S(T)-K)_+ & , S(t) < L,\ \forall t \leq T,\ \text{i.e. } \tau = T, \\ e^{-r\tau}R & , \tau < T, \end{cases} \tag{14}$$

where the constant K is the strike price, T is the expiration time, R is a prescribed cash rebate and $r > 0$ is a constant risk-free annual interest rate.

Let us assume that the cash rebate is zero, i.e. $R = 0$. Hence, the second part of the discounted payoff is zero. For the risk neutral price $C(0)$ we obtain

$$\begin{aligned} C(0) &= e^{-rT}E^Q((S(T)-K)_+ \cdot I_{\{\sup_{0\leq t\leq T} S(t) < L\}}) \\ &= e^{-rT}E^Q(\max\{S(T)-K, 0\} \cdot I_{\{\sup_{0\leq t\leq T} S(t) < L\}}), \end{aligned}$$

where I is the indicator function. If we replace the stock price by (11), we obtain

$$C(0) = e^{-rT}E^Q(\max\{S(0)e^{Z(T)} - K, 0\} \cdot I_{\{S(0)\cdot \sup_{0\leq t\leq T} e^{Z(t)} < L\}}). \tag{15}$$

We are concerned with discrete monitored barrier options, so the evaluation of the stock price $S(t)$ should be made at discrete time steps $0 = t_0 < t_1 < t_2 < \ldots < t_s = T$. For simplicity, we focus on regular time intervals, $\Delta t = t_i - t_{i-1}$. We note that

$$X_i = Z(t_i) - Z(t_{i-1}) = Z(t_{i-1} + \Delta t) - Z(t_{i-1}) \sim Z(\Delta t), \quad i = 1, \ldots, s,$$

are independent and identically distributed NIG random variables with the same distribution as $Z(\Delta t)$.

Dropping the discounted factor from the risk-neutral option price, we get the expected payoff under the Esscher transform measure of the Up-and-Out barrier call option

$$\begin{aligned} &E^Q(\max\{S(0)e^{Z(T)} - K, 0\} \cdot I_{\{S(0)\cdot e^{\sup_{0\leq t\leq T} Z(t)} < L\}}) = \\ &= E((S(0)e^{\sum_{i=1}^s X_i} - K)_+ \cdot I_{\{S(0)\cdot e^{\max_{1\leq k\leq s}\{0, \sum_{i=1}^k X_i\}} < L\}}). \end{aligned} \tag{16}$$

Our purpose is to evaluate the expected payoff (16). For this, we rewrite the expectation (16) as a multidimensional integral on $\mathbb{R}^s$

$$I = \int_{\mathbb{R}^s} \underbrace{\left(S(0)e^{\sum_{i=1}^s x^{(i)}} - K\right)_+ \cdot I_{\{S(0)\cdot e^{\max_{1\le k\le s}\{0,\sum_{i=1}^k x^{(i)}\}}<L\}}}_{E(x)} dH(x) = \int_{\mathbb{R}^s} E(x)dH(x), \tag{17}$$

where $H(x) = \prod_{i=1}^s H_i(x^{(i)}),\ \forall x = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s$, and $H_i(x^{(i)})$ denotes the distribution function of the so-called log returns induced by $Z(t_1)$, with the corresponding density function $h_i(x^{(i)})$. These log increments are independent and NIG distributed, with probability density function

$$f_{NIG}(x; \mu, \beta, \alpha, \delta) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)\right) \frac{\delta K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}} \tag{18}$$

where $K_1(x)$ denotes the modified Bessel function of third type of order 1 (see [25]).

In order to approximate the integral (17), we have to transform it to an integral on $[0,1]^s$. We can do this using an integral transformation, as follows.

We first consider the family of independent double-exponential distributions with the same parameter $\lambda > 0$, having the cumulative distribution functions $H_{\lambda,i} : \mathbb{R} \to (0,1),\ i = 1, \ldots, s$,

$$H_{\lambda,i}(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & , x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & , x \ge 0, \end{cases} \tag{19}$$

and the inverses $H_{\lambda,i}^{-1} : (0,1) \to \mathbb{R},\ i = 1, \ldots, s$, given by

$$H_{\lambda,i}^{-1}(x) = \begin{cases} \frac{1}{\lambda}\log(2x) & , 0 < x < \frac{1}{2} \\ -\frac{1}{\lambda}\log(2-2x) & , \frac{1}{2} \le x < 1. \end{cases} \tag{20}$$

Next, we consider the substitutions $x^{(i)} = H_{\lambda,i}^{-1}(1-y^{(i)}),\ i = 1, \ldots, s$, and then take $y^{(i)} = 1 - z^{(i)},\ i = 1, \ldots, s$.

The integral (17) becomes

$$\begin{aligned} I &= \int_{[0,1]^s} \underbrace{\left(S(0)e^{\sum_{i=1}^s H_{\lambda,i}^{-1}(z^{(i)})} - K\right)_+ \cdot I_{\{S(0)\cdot e^{\max_{1\le k\le s}\{0,\sum_{i=1}^k H_{\lambda,i}^{-1}(z^{(i)})\}}<L\}}}_{f(z)} dG(z) \\ &= \int_{[0,1]^s} f(z)dG(z), \end{aligned} \tag{21}$$

where $G : (0,1)^s \to [0,1]$, defined by

$$G(z) = \prod_{i=1}^s (H_i \circ H_{\lambda,i}^{-1})(z^{(i)}),\ \forall z = (z^{(1)}, \ldots, z^{(s)}) \in (0,1)^s, \tag{22}$$

is a distribution function on $(0,1)^s$, with independent marginals $G_i = H_i \circ H_{\lambda,i}^{-1}$, $i = 1,\ldots,s$.

In the case of a Double Knock-Out barrier call option, the option is valid only as long as the underlying asset remains above a lower barrier l and bellow an upper barrier L, until maturity. If the asset price touches either the upper or the lower barrier, then the option is knocked out worthless (zero payoff). Reasoning and modeling in a similar way, we need to estimate the following integral:

$$\begin{aligned} I &= \int_{[0,1]^s} \underbrace{f(z)\cdot I_{\{S(0)\cdot e^{\min_{1\le k\le s}\{0,\sum_{i=1}^k H_{\lambda,i}^{-1}(z^{(i)})\}}>l\}}}_{p(z)} dG(z) \\ &= \int_{[0,1]^s} p(z)dG(z), \end{aligned} \tag{23}$$

where function f is defined in (21) and $G:(0,1)^s \to [0,1]$, defined by

$$G(z) = \prod_{i=1}^{s} (H_i \circ H_{\lambda,i}^{-1})(z^{(i)}),\ \forall z = (z^{(1)},\ldots,z^{(s)}) \in (0,1)^s, \tag{24}$$

is a distribution function on $(0,1)^s$, with independent marginals $G_i = H_i \circ H_{\lambda,i}^{-1}$, $i = 1,\ldots,s$.

5. Numerical experiments

In the following, we compare numerically our combined method with the MC method. As a measure of comparison, we use the mean square error (MSE) produced by these methods in the approximation of the integrals (21) and (23). Next we present the estimates for the Up-and-Out barrier call option (for the Double Knock-Out barrier call option just replace function f with function p).

5.1. The MC and RSNU estimates

The MC estimate is defined as follows:

$$\bar{I}_{MC} = \frac{1}{NM}\sum_{k=1}^{NM} f(x_k), \tag{25}$$

where $x_k = (x_k^{(1)},\ldots,x_k^{(s)})$, $k = 1,\ldots,NM$, are independent identically distributed random points on $[0,1]^s$, with the common distribution function G defined in (22).

In order to generate such a point x_k, we proceed as follows. We first generate a random point $\alpha_k = (\alpha_k^{(1)}, \ldots, \alpha_k^{(s)})$, where the component $\alpha_k^{(i)}$ is a point uniformly distributed on $[0,1]$, for $i = 1, \ldots, s$. Then, for each component $\alpha_k^{(i)}$, $i = 1, \ldots, s$, we apply the inversion method [4] and we obtain that $G_i^{-1}(\alpha_k^{(i)}) = (H_{\lambda,i} \circ H_i^{-1})(\alpha_k^{(i)})$ is a point with the distribution function G_i. As the s-dimensional distribution with the distribution function G has independent marginals, it follows that $x_k = (G_1^{-1}(\alpha_k^{(1)}), \ldots, G_s^{-1}(\alpha_k^{(s)}))$ is a point with the distribution function G on $[0,1]^s$. We notice that we need to know the inverse of the distribution function of a NIG distributed random variable or, at least an approximation of it. As the inverse function is not explicitly known, an approximation of it is needed in our simulations. In order to obtain an approximation of the inverse, we use the Matlab function "niginv" as implemented by R. Werner, based on a method proposed in [25].

In what follows, we apply our combined method to estimate the integral (21). In order to do this, we need to populate the space Ω. For this, we first generate a set A that contains the first 20 prime numbers

$$A = \{2, 3, 5, 7, \ldots, 71\}.$$

Next, we construct all the subsets with s elements of the set A. There are $r = C_{20}^s$ such subsets of A. For each subset $A_i = \{p_{i,1}, \ldots, p_{i,s}\}$, we consider the SQRT point set $\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{N,i})$, defined by

$$\alpha_{k,i} = (\{k\sqrt{p_{i,1}}\}, \ldots, \{k\sqrt{p_{i,s}}\}), \qquad k = 1, \ldots, N.$$

The SQRT point sets α_i, $i = 1, \ldots, r$, are with low discrepancy in $[0,1]^s$ ([24]).

Further, we construct the space Ω of point sets with low G-discrepancy in $[0,1]^s$, $\Omega = \{\beta_1, \ldots, \beta_r\}$, where β_i, $i = 1, \ldots, r$, is of the form

$$\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i}),$$

with $\beta_{k,i} = \left(\beta_{k,i}^{(1)}, \ldots, \beta_{k,i}^{(s)}\right) \in [0,1]^s$, $k = 1, \ldots, N$.

An arbitrary point set β_i, $i = 1, \ldots, r$, is obtained from the point set α_i, using the Hlawka-Mück method ([11, 12]). This method is based on the following result.

Theorem 13. *([10]) Consider a distribution on $[0,1]^s$, with distribution function G and density function $g(u) = \prod_{j=1}^s g_j(u^{(j)})$, $\forall u = \left(u^{(1)}, \ldots, u^{(s)}\right) \in [0,1]^s$. Assume that the marginal density functions g_j, $j = 1, \ldots, s$, are continuous on $[0,1]$. Furthermore, assume that $g_j(t) \neq 0$, for almost every $t \in [0,1]$ and for all $j = 1, \ldots, s$. Denote by $M_g = \sup_{u \in [0,1]^s} g(u)$. Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a set of points in $[0,1]^s$. Generate the set of points $\beta = (\beta_1, \ldots, \beta_N)$, with*

$$\beta_k^{(j)} = \frac{1}{N} \sum_{r=1}^N \left[1 + \alpha_k^{(j)} - G_j\left(\alpha_r^{(j)}\right)\right] = \frac{1}{N} \sum_{r=1}^N 1_{[0,\alpha_k^{(j)}]}\left(G_j\left(\alpha_r^{(j)}\right)\right),$$

for all $k = 1, \ldots, N$ and all $j = 1, \ldots, s$, where $[a]$ denotes the integer part of a. Then the generated set of points has a G-discrepancy of

$$D_{N,G}(\beta_1, \ldots, \beta_N) \leq (2 + 6sM_g)D_N(\alpha_1, \ldots, \alpha_N).$$

Next, we select the integers $i_1, \ldots, i_M$ at random from the uniform distribution on $\{1, \ldots, r\}$ and consider the corresponding point sets with low G-discrepancy $\beta_{i_1}, \ldots, \beta_{i_M}$.

We calculate the following estimate:

$$\bar{I}_{RSNU} = \frac{\sum_{l=1}^{M} \left(\frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i_l}) \right)}{M}. \tag{26}$$

In our numerical experiments, we consider that the parameters of the NIG-distributed log-returns under the equivalent martingale measure given by the Esscher transform are the ones that are given in [14]:

$$\mu = 0.00079 * 5, \ \beta = -15.1977, \ \alpha = 136.29, \ \delta = 0.0059 * 5. \tag{27}$$

We notice that these parameters are relevant for daily observed stock price log-returns (see [33]). As the class of NIG distributions is closed under convolution, we can derive weekly stock prices by using a factor of 5 for the parameters μ and δ.

5.2. Up-and-Out barrier options

We suppose that the initial stock price is $S(0) = 100$, the strike price is $K = 100$, the barrier price is $L = 105$ and the risk-free annual interest rate is $r = 3.75\%$. We choose the parameter of the double-exponential distribution $\lambda = 95.2271$.

The barrier option is sampled at weekly time intervals. We also let the option to have maturities of $s = 5$ weeks. Hence, our problem is a 5 multidimensional integral, over the payoff function.

Because no analytical solution is known for this type of options, the "exact" price is obtained as the average of 30 MC simulations, with $N = 500000$ for the initial integral (17).

The quality of an estimator $\overline{\theta}$, for a given sample size, is measured by its mean squared error $\mathrm{MSE}(\overline{\theta})$, which is the expected value of the squared difference between the estimator $\overline{\theta}$ and the true value I of the parameter we estimate:

$$\mathrm{MSE}(\overline{\theta}) = E[(\overline{\theta} - I)^2].$$

We are going to compare the MC and RSNU estimators in terms of their mean square errors, MSE($\overline{I}_{MC}$) and MSE($\overline{I}_{RSNU}$). We fix M at 100 and we only increase N from 100 to 1000, with a step of 225.

For a given N (N = 100, 325, 550, 775, 1000), we make 30 independent runs (the problem is simulated 30 times), using the MC and the combined method.

Table 1 displays the value of MN, the mean square error of thirty option estimates computed using MC method and the mean square error of thirty option estimates computed using RSNU method.

MN	Mean square error MC	Mean square error RSNU
10000	1.50×10^{-4}	9.43×10^{-5}
32500	3.84×10^{-5}	5.58×10^{-5}
55000	4.67×10^{-5}	4.25×10^{-5}
77500	2.99×10^{-5}	2.73×10^{-5}
100000	1.97×10^{-5}	9.70×10^{-6}

Table 1: Up-and-Out Barrier call option: Simulation results.

The numerical results for the Up-and-Out Barrier call option, indicate that in most of the cases, the mean square error produced by our combined method is smaller than that produced by the MC method.

5.3. Double Knock-Out barrier options

In the following we take: the initial stock price $S(0) = 100$, the strike price $K = 100$, the lower barrier price $l = 97$, the upper barrier price $L = 105$ and the risk-free annual interest rate $r = 3.75\%$. We choose the parameter of the double-exponential distribution $\lambda = 95.2271$.

The barrier option is sampled at weekly time intervals. We also let the option to have maturities of $s = 5$ weeks. Hence, our problem is a 5 multidimensional integral, over the payoff function.

The "exact" price is obtained as the average of 30 MC simulations, with $N = 500000$ for the initial integral (23).

We are going to compare the MC and RSNU estimators in terms of their mean square errors.

We fix M at 100 and we only increase N from 100 to 1000, with a step of 225.

For a given N (N = 100, 325, 550, 775, 1000), we make 30 independent runs (the problem is simulated 30 times), using the MC and the combined method.

Table 2 displays the value of MN, the mean square error of thirty option estimates computed using MC method and the mean square error of thirty option estimates computed using RSNU method.

MN	Mean square error MC	Mean square error RSNU
10000	1.58×10^{-4}	9.52×10^{-5}
32500	4.07×10^{-5}	5.52×10^{-5}
55000	4.65×10^{-5}	3.55×10^{-5}
77500	2.93×10^{-5}	2.63×10^{-5}
100000	2.00×10^{-5}	8.63×10^{-6}

Table 2: Double Knock-Out barrier call option: Simulation results.

From Table 2, we notice that in almost all cases, the mean square error produced by our combined method is smaller than that of the MC method.

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