Acta Universitatis Apulensis No. 28/2011
ISSN: 1582-5329 pp. 45-60

SHARP FUNCTION ESTIMATE AND BOUNDEDNESS ON MORREY SPACES FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR

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Abstract: In this paper, we prove the sharp estimates for the multilinear commutator related to the Marcinkiewicz operator. By using the sharp estimates, we obtain the boundedness of the multilinear commutator on Morrey spaces.

2000 *Mathematics Subject Classification*: 42B20, 42B25.

1. Introduction

As the development of singular integral operators, their commutators have been well studied(see [1][3-5][10-12]). Let T be the Calderón-Zygmund singular integral operator. A classical result of Coifman, Rocherberg and Weiss (see [3]) state that commutator $[b,T](f) = T(bf) - bT(f)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [10-12], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The Marcinkiewicz operator is an important operator in harmonic analysis(see [6-8]). As the Morrey space may be considered as an extension of Lebesgue space, it is natural and important to study the boundedness of multilinear commutator related to the Marcinkiewicz operator on the Morrey Spaces. The purpose of this paper has two, first, we establish a sharp estimate for the multilinear commutator related to the Marcinkiewicz operator, and second, we prove the boundedness for the multilinear commutator related to the Marcinkiewicz operator on the generalized Morrey spaces by using the sharp estimate.

2. Preliminaries and Theorem

Let us introduce some notations(see [2][4][15]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f on R^n and a cube Q, let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is well-known that (see [4])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - C| dy.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^{\infty}(R^n)$ and define $||f||_{BMO} = ||M^{\#}(f)||_{L^{\infty}}$. It has been known that (see [4])

$$||f - f_{2^k Q}||_{BMO} \leq Ck||f||_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy.$$

We write that $M_p(f) = (M(|f|^p))^{\frac{1}{p}}$ for $0 < p < \infty$. For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } \ k \geq 2.$$

Let $0 < \delta < n$, $0 < r < \infty$, set

$$M_{r,\delta}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\delta r/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f,

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$

and the maximal function associated to Φ by

$$M_{\Phi}(f)(x) = \sup_{x \in Q} ||f||_{\Phi,Q}.$$

The Young functions to be using in this paper are $\Phi(t) = t(1 + logt)^r$ and $\tilde{\Phi}(t) = exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$ and $|| \cdot ||_{expL^{1/r},Q}$, $M_{expL^{1/r}}$. Following [12][15], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|} \int_Q |f(y)g(y)| dy \leq ||f||_{\Phi,Q} ||g||_{\tilde{\Phi},Q}$$

and the following inequality, for $r, r_j \geq 1, j = 1, \cdots, m$ with $1/r = 1/r_1 + \cdots + 1/r_m$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$||f||_{L(logL)^{1/r},Q} \leq M_{L(logL)^{1/r}}(f) \leq CM_{L(logL)^m}(f) \leq CM^{m+1}(f),$$

$$||b - b_Q||_{expL^r,Q} \leq C||b||_{BMO},$$

$$|b_{2^{k+1}Q} - b_{2Q}| \leq Ck||b||_{BMO}.$$

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements and $\sigma(i) < \sigma(j)$ when $i < j$. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = \prod_{i=1}^{j} b_{\sigma(i)}$ and $||\vec{b}_\sigma||_{BMO} = \prod_{i=1}^{j} ||b_{\sigma(i)}||_{BMO}$.

We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$(see [4]), that is

$$A_1 = \{w : M(w)(x) \leq Cw(x), a.e.\}$$

and

$$A_p = \left\{ w : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty \right\}, \ 1 < p < \infty.$$

Throughout this paper, φ will denote a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for} \quad t \geq 0.$$

Let w be a weight function on R^n(that is w is a non-negative locally integrable function on R^n) and f be a locally integrable function on R^n. Define, for $1 \leq p < \infty$,

$$||f||_{L^{p,\varphi}(w)} = \sup_{x \in R^n, \ d > 0} \left(\frac{1}{\varphi(d)} \int_{B(x,d)} |f(y)|^p w(y)dy \right)^{1/p},$$

where $B(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized weighted Morrey spaces is defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey spaces (see [9][13][14]).

In this paper, we will study some multilinear commutators as following.

Definition. *Let* $0 < \delta < n$, $0 < \gamma \le 1$ *and* Ω *be homogeneous of degree zero on* R^n *such that* $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. *Assume that* $\Omega \in Lip_\gamma(S^{n-1})$, *that is there exists a constant* $M > 0$ *such that for any* $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x - y|^\gamma$. *The Marcinkiewicz multilinear commutator is defined by*

$$\mu_{\Omega,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_{t,\delta}^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{t,\delta}^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] f(y)dy.$$

Set

$$F_{t,\delta}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy,$$

we also define that

$$\mu_{\Omega,\delta}(f)(x) = \left(\int_0^\infty |F_{t,\delta}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [16]).

Let H *be the space* $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t^3}\right)^{1/2} < \infty\right\}$. *Then, it is clear that*

$$\mu_{\Omega,\delta}(f)(x) = ||F_{t,\delta}(f)(x)|| \text{ and } \mu_{\Omega,\delta}^{\vec{b}}(f)(x) = ||F_{t,\delta}^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $\mu_{\Omega,\delta}^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1][5-8][10-12][16]). Our main purpose is to establish the sharp inequality and boundedness on the Morrey spaces for the multilinear commutator.

We shall prove the following theorems.

Theorem 1. *Let* $b_j \in BMO$ *for* $j = 1, \cdots, m$. *Then for any* $1 < r < \infty$ *and* $0 < r_0 < 1$, *there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(R^n)$ *and any* $\tilde{x} \in R^n$,

$$\left(\mu_{\Omega,\delta}^{\vec{b}}(f)\right)_{r_0}^{\#}(\tilde{x}) \le C\left(||\vec{b}||_{BMO} M_{r,\delta}(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma\in C_j^m} ||\vec{b}_\sigma||_{BMO} M^{m+1}(\mu_{\Omega,\delta}^{\vec{b}_{\sigma^c}}(f))(\tilde{x})\right).$$

Theorem 2. *Let $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, $0 < D < 2^n$, $w \in A_1$. Then there exists a constant $C > 0$ such that for any $f \in L^{p,\varphi}(R^n, w)$,*

$$||\mu^{\vec{b}}_{\Omega,\delta}(f)||_{L^{q,\varphi}(w)} \leq C||f||_{L^{p,\varphi}(w)}.$$

3. Proofs of Theorems

To prove the theorem, we need the following lemmas.

Lemma 1.(see [6][8]) *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $w \in A_1$. Then $\mu_{\Omega,\delta}$ is bounded from $L^p(w)$ to $L^q(w)$.*

Lemma 2.(see [4]) *Let $0 < \delta < n$, $1 \leq r < p < n/\delta$, $1/q = 1/p - \delta/n$ and $w \in A_1$. Then $M_{r,\delta}$ is bounded from $L^p(w)$ to $L^q(w)$.*

Lemma 3. *Let $1 < r < \infty$, $\sigma \in C^m_k$ with $k \leq m$ and $m \in N$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leq C \prod_{j=1}^k ||b_j||_{BMO},$$

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \leq C \prod_{j=1}^k ||b_j||_{BMO},$$

$$\frac{1}{|Q|}\int_Q |(b(y) - (b_j)_Q)_\sigma| dy \leq C||b_\sigma||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q |(b(y) - (b_j)_Q)_\sigma|^r dy\right)^{1/r} \leq C||b_\sigma||_{BMO}.$$

In fact, we just need to choose $p_j > 1$ and $q_j > 1$, where $1 \leq j \leq k$, such that $1/p_1 + \cdots + 1/p_k = 1$ and $r/q_1 + \cdots + r/q_k = 1$. After that, using the Hölder's inequality with exponent $1/p_1 + \cdots + 1/p_k = 1$ and $r/q_1 + \cdots + r/q_k = 1$ respectively, we may get the results.

Lemma 4. *Let $1 < p < \infty$, $0 < D < 2^n$ and $w \in A_1$. Then, for $f \in L^{p,\varphi}(R^n, w)$,*

(a) $||M(f)||_{L^{p,\varphi}(w)} \leq C||f^{\#}||_{L^{p,\varphi}(w)}$;

(b) $||M_q(f)||_{L^{p,\varphi}(w)} \leq C||f||_{L^{p,\varphi}(w)}$ *for $1 < q < p$.*

Proof. *(a). Let $f \in L^{p,\varphi}(R^n, w)$. Note that the following inequality (see [4], p.410): for any $u \in A_1$,*

$$\int_{R^n} |M(f)(y)|^p u(y) dy \leq C \int_{R^n} |f^{\#}(y)|^p u(y) dy,$$

For a ball $B = B(x,d) \subset R^n$, we get

$$
\begin{aligned}
& \int_B |M(f)(y)|^p w(y)dy \\
\leq\ & \int_{R^n} |M(f)(y)|^p M(w\chi_B)(y)dy \\
\leq\ & C\int_{R^n} |f^{\#}(y)|^p M(w\chi_B)(y)dy \\
=\ & C\left[\int_B |f^{\#}(y)|^p M(w\chi_B)(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB} |f^{\#}(y)|^p M(w\chi_B)(y)dy\right] \\
\leq\ & C\left[\int_B |f^{\#}(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB} |f^{\#}(y)|^p \frac{w(B)}{|2^{k+1}B|}dy\right] \\
\leq\ & C\left[\int_B |f^{\#}(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B} |f^{\#}(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}dy\right] \\
\leq\ & C\left[\int_B |f^{\#}(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B} |f^{\#}(y)|^p \frac{w(y)}{2^{nk}}dy\right] \\
\leq\ & C||f^{\#}||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty} 2^{-nk}\varphi(2^{k+1}d) \\
\leq\ & C||f^{\#}||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty} (2^{-n}D)^k\varphi(d) \\
\leq\ & C||f^{\#}||^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}
$$

thus,

$$||M(f)||_{L^{p,\varphi}(w)} \leq C||f^{\#}||_{L^{p,\varphi}(w)}.$$

(b). Let $f \in L^{p,\varphi}(R^n, w)$. Note that $1 < q < p$ and for any $u \in A_1$,

$$\int_{R^n} |M_q(f)(y)|^p u(y)dy \leq C\int_{R^n} |f(y)|^p u(y)dy.$$

For a ball $B = B(x,d) \subset R^n$, a similar argument as in the proof of (a), we get

$$
\begin{aligned}
& \int_B |M_q(f)(y)|^p w(y)dy \\
\leq\ & \int_{R^n} |M_q(f)(y)|^p M(w\chi_B)(y)dy \\
\leq\ & C\left[\int_B |f(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB} |f(y)|^p \frac{w(B)}{|2^{k+1}B|}dy\right]
\end{aligned}
$$

$$\begin{aligned}
&\leq C\left[\int_B |f(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}dy\right]\\
&\leq C\left[\int_B |f(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f(y)|^p \frac{w(y)}{2^{nk}}dy\right]\\
&\leq C||f||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(d)\\
&\leq C||f||^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus,

$$||M_q(f)||_{L^{p,\varphi}(w)} \leq C||f||_{L^{p,\varphi}(w)}.$$

Lemma 5. *Let $0 < \delta < n$, $1 \leq r < p < n/\delta$, $1/q = 1/p - \delta/n$, $0 < D < 2^n$ and $w \in A_1$. Then, for $f \in L^{p,\varphi}(R^n, w)$,*

$$||M_{r,\delta}(f)||_{L^{q,\varphi}(w)} \leq ||f||_{L^{p,\varphi}(w)}.$$

Lemma 6. *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$, $0 < D < 2^n$ and $w \in A_1$. Then, for $f \in L^{p,\varphi}(R^n, w)$,*

$$||\mu_{\Omega,\delta}(f)||_{L^{q,\varphi}(w)} \leq C||f||_{L^{p,\varphi}(w)}.$$

The proofs of two Lemmas are similar to that of Lemma 4 by Lemma 1 and 2, we omit the details.

Proof of Theorem 1. *It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:*

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |\mu^{\vec{b}}_{\Omega,\delta}(f)(\tilde{x}) - C_0|^{r_0}dx\right)^{1/r_0}\\
&\leq C\left(||\vec{b}||_{BMO}M_{r,\delta}(f)(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M^{m+1}\mu^{\vec{b}_{\sigma^c}}_{\Omega,\delta}(f))(\tilde{x})\right).
\end{aligned}$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$, we write $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$. We will consider the cases $m = 1$ and $m > 1$, and choose $C_0 = \mu_{\Omega,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)$ and $C_0 = \mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)$, respectively.

We first consider the **Case** $m = 1$. *For* $C_0 = \mu_{\Omega,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)$, *we write*

$$F^{b_1}_{t,\delta}(f)(x) = (b_1(x)-(b_1)_{2Q})F_{t,\delta}(f)(x)-F_{t,\delta}((b_1-(b_1)_{2Q})f_1)(x)-F_{t,\delta}((b_1-(b_1)_{2Q})f_2)(x),$$

then

$$\begin{aligned}
& |\mu^{b_1}_{\Omega,\delta}(f)(x) - \mu_{\Omega,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
= & \left| ||F^{b_1}_{t,\delta}(f)(x)|| - ||F_{t,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)|| \right| \\
\leq & ||F^{b_1}_{t,\delta}(f)(x) - F_{t,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)|| \\
\leq & ||(b_1(x) - (b_1)_{2Q})F_{t,\delta}(f)(x)|| + ||F_{t,\delta}((b_1 - (b_1)_{2Q})f_1)(x)|| \\
& + ||F_{t,\delta}((b_1 - (b_1)_{2Q})f_2)(x) - F_{t,\delta}((b_1 - (b_1)_{2Q})f_2)(x_0)|| \\
= & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, *we get*

$$\begin{aligned}
& \left(\frac{1}{|Q|}\int_Q |A(x)|^{r_0}dx\right)^{1/r_0} \leq \frac{1}{|Q|}\int_Q |A(x)|dx \\
= & \frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}||\mu_{\Omega,\delta}(f)(x)|dx \\
\leq & C||b_1 - (b_1)_{2Q}||_{\exp L,2Q}||\mu_{\Omega,\delta}(f)||_{L(\log L),2Q} \\
\leq & C||b_1||_{BMO}M^2(\mu_{\Omega,\delta}(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, *denoting* $r = ps$, $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, *by the boundness of* $\mu_{\Omega,\delta}$ *from* $L^p(R^n)$ *to* $L^q(R^n)$ *and Hölder's inequality with exponent*$1/s + 1/s' = 1$, *we have*

$$\begin{aligned}
& \left(\frac{1}{|Q|}\int_Q |B(x)|^{r_0}dx\right)^{1/r_0} \leq \frac{1}{|Q|}\int_Q |B(x)|dx \\
= & \frac{1}{|Q|}\int_Q [\mu_{\Omega,\delta}((b_1 - (b_1)_{2Q})f_1)(x)]dx \\
\leq & \left(\frac{1}{|Q|}\int_{R^n} [\mu_{\Omega,\delta}((b_1 - (b_1)_{2Q})f\chi_{2Q})(x)]^q dx\right)^{1/q} \\
\leq & C\frac{1}{|Q|^{1/q}}\left(\int_{R^n} |b_1(x) - (b_1)_{2Q}|^p|f(x)\chi_{2Q}(x)|^p dx\right)^{1/p} \\
\leq & C|Q|^{(-1/q)+(1/ps')+(1-\delta ps/n)/ps}\left(\frac{1}{|2Q|}\int_{2Q} |b_1 - (b_1)_{2Q}|^{ps'}dx\right)^{1/ps'} \\
& \times\left(\frac{1}{|2Q|^{1-\delta ps/n}}\int_{2Q} |f(x)|^{ps}dx\right)^{1/ps}
\end{aligned}$$

$$
\begin{aligned}
&= C|Q|^{(-1/q)+(1/ps')+(1-\delta r/n)/r}\left(\frac{1}{|2Q|}\int_{2Q}|b_1-(b_1)_{2Q}|^{ps'}dx\right)^{1/ps'}\\
&\quad\times\left(\frac{1}{|2Q|^{1-\delta r/n}}\int_{2Q}|f(x)|^r dx\right)^{1/r}\\
&\leq C||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}
$$

For $C(x)$, note that $|x_0-y|\approx|x-y|$ for $y\in Q^c$, we have

$$
\begin{aligned}
C(x) &= ||F_{t,\delta}((b_1-(b_1)_{2Q})f_2)(x)-F_{t,\delta}((b_1-(b_1)_{2Q})f_2)(x_0)||\\
&= \left(\int_0^\infty\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1-\delta}}(b_1(y)-(b_1)_{2Q})dy\right.\right.\\
&\quad\left.\left.-\int_{|x_0-y|\leq t}\frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1-\delta}}(b_1(y)-(b_1)_{2Q})dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&\leq \left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}|(b_1(y)-(b_1)_{2Q})|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\leq t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}|(b_1(y)-(b_1)_{2Q})|\,dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|\leq t}\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1-\delta}}\right|\right.\right.\\
&\quad\left.\left.\times|(b_1(y)-(b_1)_{2Q})||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\equiv I_1+I_2+I_3.
\end{aligned}
$$

By the Minkowski's inequality and Hölder's inequality with exponent $1/r'+1/r=1$,

$$
\begin{aligned}
I_1 &\leq C\int_{(2Q)^c}|(b_1(y)-(b_1)_{2Q})|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left(\int_{|x-y|\leq t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leq C\int_{(2Q)^c}|(b_1(y)-(b_1)_{2Q})|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy\\
&\leq C\int_{(2Q)^c}|(b_1(y)-(b_1)_{2Q})|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy\\
&\leq C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|(b_1(y)-(b_1)_{2Q})||x_0-y|^\delta\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|(b_1(y)-(b_1)_{2Q}||f(y)|dy\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|(b_1(y)-(b_1)_{2Q})|^{r'}dy\right)^{1/r'}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|f(y)|^{r}dy\right)^{1/r}\\
&= C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{(\delta/r'n)+\delta(1-r)/rn}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_1(y)-(b_1)_{2Q})|^{r'}dy\right)^{1/r'}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|^{1-\delta r/n}}\int_{2^{k+1}Q}|f(y)|^{r}dy\right)^{1/r}\\
&\leq C\sum_{k=1}^{\infty}k2^{-k/2}||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x})\\
&\leq C||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x});
\end{aligned}
$$

Similarly, we have $I_2\leq C||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x})$.

We now estimate I_3. *By the following inequality:*

$$
\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1-\delta}}\right|\leq C\left(\frac{|x-x_0|}{|x_0-y|^{n-\delta}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\right),
$$

we gain

$$
\begin{aligned}
I_3 &\leq C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}|\frac{|f(y)||x-x_0|}{|x_0-y|^{n-\delta}}\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\quad+C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}|\frac{|f(y)||x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_{2Q}|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy\\
&\leq C\sum_{k=1}^{\infty}k(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}||f(y)|dy\\
&\leq C\sum_{k=1}^{\infty}k(2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{(\delta/r'n)+\delta(1-r)/rn}\\
&\times\left(|\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_1(y)-(b_1)_{2Q})|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^{k+1}Q|^{1-\delta r/n}}\int_{2^{k+1}Q}|f(y)|^{r}dy\right)^{1/r}\\
&\leq C||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of the case $m=1$.

Now, we consider the case $m \geq 2$. *We write, for* $\vec{b} = (b_1, ..., b_m)$,

$$F^{\vec{b}}_{t,\delta}(f)(x) = \int_{|x-y|\leq t} \left[\prod_{j=1}^{m}(b_j(x) - b_j(y))\right] \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy$$

$$= \int_{|x-y|\leq t} \left[\prod_{j=1}^{m}((b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q}))\right] \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy$$

$$= \sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma \int_{|x-y|\leq t}(b(y)-(b)_{2Q})_{\sigma^c}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}f(y)dy$$

$$= \prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q} F_{t,\delta}(f)(x) + (-1)^m F_{t,\delta}(\prod_{j=1}^{m}(b_j(y) - (b_j)_{2Q})f)(x)$$

$$+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma F^{\vec{b}_{\sigma^c}}_{t,\delta}(f)(x),$$

thus, recall that $C_0 = \mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)$,

$$|\mu^{\vec{b}}_{\Omega,\delta}(f)(x) - \mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)|$$

$$\leq ||F^{\vec{b}}_{t,\delta}(f)(x) - F_{t,\delta}(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)||$$

$$\leq ||\prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})F_{t,\delta}(f)(x)|| + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_{2Q})_\sigma F^{\vec{b}_{\sigma^c}}_{t,\delta}(f)(x)||$$

$$+ ||F_{t,\delta}(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_1)(x)||$$

$$+ ||F_{t,\delta}(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x) - F_{t,\delta}(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x_0)||$$

$$= S_1(x) + S_2(x) + S_3(x) + S_4(x).$$

For $S_1(x)$, *we get*

$$\left(\frac{1}{|Q|}\int_Q |S_1(x)|^{r_0}dx\right)^{1/r_0} \leq \frac{1}{|Q|}\int_Q |S_1(x)|dx$$

$$= \frac{1}{|Q|}\int_Q |\prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q}||\mu_{\Omega,\delta}(f_1)(x)|dx$$

$$\begin{aligned} &\le C||\prod_{j=1}^m (b_j-(b_j)_{2Q})||_{\exp L^{r_j},2Q}||\mu_{\Omega,\delta}(f)||_{L(\log L)^{1/r},2Q} \\ &\le C\prod_{j=1}^m ||b_j||_{BMO}M^{m+1}(\mu_{\Omega,\delta}(f))(\tilde{x}). \end{aligned}$$

For $S_2(x)$, by Lemma 3, we get

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |S_2(x)|^{r_0}dx\right)^{1/r_0} \le \frac{1}{|Q|}\int_Q S_2(x)dx \\ = &\frac{1}{|Q|}\int_Q \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2Q})_\sigma F_{t,\delta}^{\vec{b}_{\sigma^c}}(f)(x)||dx \\ \le &\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(x)-(b)_{2Q})_\sigma||\mu_{\Omega,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|dx \\ \le &C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b-(b)_{2Q})_\sigma||_{\exp L^{r_j},2Q}||\mu_{\Omega,\delta}^{\vec{b}_{\sigma^c}}(f)||_{L(\log L)^{1/r},2Q} \\ \le &C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||\vec{b}_\sigma||_{BMO}M^{m+1}(\mu_{\Omega,\delta}^{\vec{b}_{\sigma^c}}(f)(\tilde{x})). \end{aligned}$$

For $S_3(x)$, denoting $r=ps$, $1<p<q<n/\delta$, $1/q=1/p-\delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with exponent $1/s+1/s'=1$, we have

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |S_3(x)|^{r_0}dx\right)^{1/r_0} \le \frac{1}{|Q|}\int_Q |S_3(x)|dx \\ = &\frac{1}{|Q|}\int_Q ||F_{t,\delta}(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_1)(x)||dx \\ \le &\left(\frac{1}{|Q|}\int_{R^n} |\mu_{\Omega,\delta}(\prod_{j=1}^m (b_j-(b_j)_{2Q})f\chi_{2Q})(x)|^q dx\right)^{1/q} \\ \le &C\frac{1}{|Q|^{1/q}}\left(\int_{R^n} |\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})|^p|f(x)\chi_{2Q}(x)|^p dx\right)^{1/p} \\ \le &C\frac{1}{|Q|^{1/q}}\left(\int_{2Q} |\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})|^{ps'}dx\right)^{1/ps'}\left(\int_{2Q}|f(x)|^{ps}dx\right)^{1/ps} \end{aligned}$$

$$
\begin{aligned}
&\le C|Q|^{(-1/q)+(1/ps')+(1-(\delta ps/n)/ps)}\left(\frac{1}{|2Q|}\int_{2Q}|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})|^{ps'}dx\right)^{1/ps'}\\
&\quad\times\left(\frac{1}{|2Q|^{1-\delta ps/n}}\int_{2Q}|f(x)|^{ps}dx\right)^{1/ps}\\
&\le C||\vec{b}||_{BMO}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}
$$

For $S_4(x)$, similar to the proof of $C(x)$ in the case $m=1$, we get

$$
\begin{aligned}
S_4(x) &= ||F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x)-F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x_0)||\\
&= \left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1-\delta}}\left[\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right]dy\right.\right.\\
&\quad-\left.\left.\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1-\delta}}\left[\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right]dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&\le \left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1-\delta}}\right|\right.\right.\\
&\quad\times\left.\left.\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\equiv V_1+V_2+V_3,
\end{aligned}
$$

thus

$$
\begin{aligned}
V_1 &\le C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)|}{|x-y|^{n-1-\delta}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy\\
&\leq C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| |x_0-y|^\delta \frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| |f(y)|dy\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right|^{r'} dy\right)^{1/r'}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r}\\
&= C\sum_{k=1}^\infty 2^{-k/2}|2^{k+1}Q|^{(\delta/r'n)+\delta(1-r)/rn}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right|^{r'} dy\right)^{1/r'}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|^{1-\delta r/n}}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r}\\
&\leq C||\vec{b}||_{BMO}M_{r,\delta}(f)(\tilde{x});
\end{aligned}
$$

Similarly, we have $V_2 \leq C||\vec{b}||_{BMO}M_{r,\delta}(f)(\tilde{x})$. *For* V_3, *we gain*

$$
\begin{aligned}
V_3 &\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)||x-x_0|}{|x_0-y|^{n-\delta}}\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\quad+C\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)||x-x_0|^\gamma}{|x_0-y|^{n-1-\delta+\gamma}}\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leq C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy\\
&\leq C\sum_{k=1}^\infty (2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| |f(y)|dy\\
&\leq C\sum_{k=1}^\infty (2^{-k}+2^{-k\gamma})\prod_{j=1}^m ||b_j||_{BMO}M_{r,\delta}(f)(\tilde{x})\\
&\leq C||\vec{b}||_{BMO}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}
$$

This completes the total proof of Theorem 1.

Proof of Theorem 2. *Now we first consider the case m=1, we obtain, by Lemma 5 and 6,*

$$\begin{aligned} ||\mu_{\Omega,\delta}^{b_1}(f)||_{L^{q,\varphi}(w)} &\leq ||M(\mu_{\Omega,\delta}^{b_1}(f))||_{L^{q,\varphi}(w)} \leq ||(\mu_{\Omega,\delta}^{b_1}(f))_{r_0}^{\#}||_{L^{q,\varphi}(w)} \\ &\leq C||b_1||_{BMO}(||M_{r,\delta}(f)||_{L^{q,\varphi}(w)} + ||M^2(\mu_{\Omega,\delta}(f))||_{L^{q,\varphi}(w)}) \\ &\leq C||b_1||_{BMO}(||f||_{L^{p,\varphi}(w)} + ||\mu_{\Omega,\delta}(f)||_{L^{q,\varphi}(w)}) \\ &\leq C||b_1||_{BMO}||f||_{L^{p,\varphi}(w)} \leq C||f||_{L^{p,\varphi}(w)}. \end{aligned}$$

When $m \geq 2$, we may get the conclusion of Theorem by induction. This completes the proof of Theorem 2.

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